THE LIT-ONLY σ-GAME: SOME MATHEMATICS BEHIND A PUZZLE FOR KIDS

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Abstract. The lit-only σ-game is a nondeterministic linear dynamical process whose local transition rule is specified by a digraph. The player of the game can choose among several possible transitions at each step and aim to achieve certain global phase transition. This essay tries to popularize this game to kids for their fun. We also give readers a taste for some possible mathematics behind the game. The paper basically contains no proofs and many ideas are not presented in their full details. We invite the readers to venture out into the world of the lit-only σ-game and other discrete-time graph processes.

Keywords: binary field, line graph, lit-only group, nuclei, phase space, quadratic form, reachability, transvection.

Contents

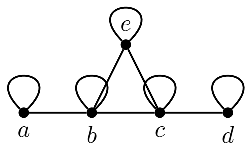


Figure 1.1. The Nandi (bull) graph.

1. Appetizer

Jaap Scherphuis maintains a webpage [1] on various puzzles. We note that a subpage there is entitled "The Mathematics of Lights Out" [2]. Then, what is this lights out game and can this puzzle really lead to any interesting mathematics?

Different players may choose different names for the same puzzle and sometimes they may adopt game rules which are not exactly the same. The lights out game and its lit-only version will be introduced as the σ-game and the lit-only σ-game in the following. Here is a rough description of the σ-game as adapted from what is told by Andries E. Brouwer [3]:

> We are given a finite graph, with a light at each vertex that can be on or off. Pressing the button at a vertex switches the state of the lights at its neighbours. The question is whether all lights can be switched off given an arbitrary starting position.

In Fig. 1.1, we display the Nandi (bull) graph where every vertex is attached a loop: You may think of a and d as the horns, b and c as the eyes, and e as the nose. In Fig. 1.2 we are displaying the process of moving from the all-lights-on configuration to the all-lights-off configuration by pressing vertices e, b and c in that order when playing the σ-game on the Nandi graph. Note that here we use a disk to indicate a vertex in the on state and a circle to refer to a vertex in the off state.

The lit-only σ-game on a graph is the variant of the σ-game in which one is not allowed to toggle any vertex when it is off. Among the three moves in Fig. 1.2, which are valid pushings in the σ-game, only the middle one is not allowed in the lit-only σ-game. You can tell from Fig. 1.2 that we use $\rightarrow$ for a move in the lit-only σ-game (and also in the σ-game) and we write $\rightarrow_\sigma$ for a pushing in the σ-game (which may be invalid in the lit-only σ-game). Can we go from the all-lights-on configuration to the all-lights-off configuration when playing the lit-only σ-game on the Nandi graph? To turn off all-lights, we have to press a vertex an

[1] http://www.jaapsch.net/puzzles/lights.htm

[2] http://www.jaapsch.net/puzzles/lomath.htm

[3] http://www.win.tue.nl/~aeb/ca/madness/madrect.html

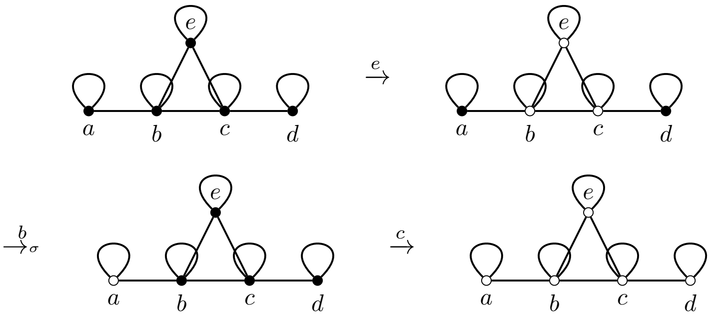


Figure 1.2. Playing the σ-game on the Nandi graph.

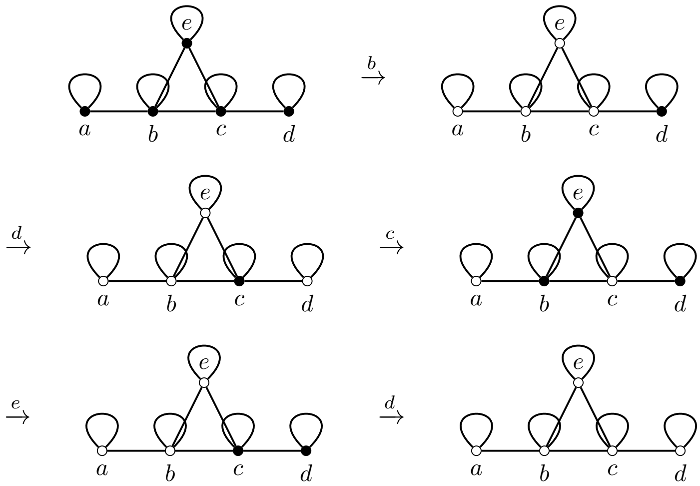


Figure 1.3. Playing the lit-only σ-game on the Nandi graph.

odd number of times if it is the nose or an eye and we have to press a vertex an even number of times if it is a horn. But, after pressing any eye or the nose, the eyes and the nose will be all off and so it seems that we do not have any way to press every one of them an odd number of times. Does it imply the impossibility of moving from the all-lights-on to the all-lights-off in the lit-only σ-game on the Nandi graph? Surely, you will not be misled by this plausible argument and, if you have a try for a little while, you will find a required path from the all-lights-on to the all-lights-off, say the one shown in Fig. 1.3.

2. Tiffin

Playing on the Nandi graph may only be interesting as a kindergarten game. To entertain elementary school students, let us share with you some more secrets about the σ-game and its lit-only relative. Before doing that, we change gears and build on a definition of our game in a more mathematical way.

A *digraph* D consists of its nonempty vertex set V_D and its arc set $\mathrm{A}_D \subseteq \mathrm{V}_D \times \mathrm{V}_D$. We say that D is *empty* when the arc set A_D is the empty set. An element $(u, v) \in \mathrm{V}_D \times \mathrm{V}_D$ will sometimes be conveniently recorded as uv. We reserve the notation L_D for the set of *loops* in D, namely $\mathrm{L}_D := \{v \in \mathrm{V}_D : vv \in \mathrm{A}_D\}$. For each $v \in \mathrm{V}_D$, we write $\mathrm{N}^+_D(v)$ for $\{w : vw \in \mathrm{A}_D\}$ and call it the set of *out-neighbours* of v in D. A digraph D is *symmetric* provided, for all $u, v \in \mathrm{V}_D$, either both uv and vu are arcs in D or none of them is an arc in D. A digraph is called a *graph* if it is symmetric and called *asymmetric* otherwise. A graph D can be thought of as a pair consisting of its vertex set V_D and edge set $\mathrm{E}_D \subseteq \binom{\mathrm{V}_D}{1} \cup \binom{\mathrm{V}_D}{2}$ where $\{u, v\} \in \mathrm{E}_D$ if and only if $uv \in \mathrm{A}_D$. For a graph D, we often simplify the notation $\mathrm{N}^+_D(v)$ as $\mathrm{N}_D(v)$ and call it the set of *neighbours* of v in D. A digraph is *strongly connected* if, for any two vertices of it, say u and v, there exist a nonnegative integer k and a sequence of vertices $u = u_0, u_1, \ldots, u_k = v$ such that $u_{i-1}u_i$ is an arc of the digraph for every positive integer i not greater than k. For a graph, we often simply say that it is *connected* to mean that it is strongly connected. The set of all digraphs D on a vertex set V can be identified with the set of single variable functions β from V to $G := 2^V$ [53]: Given D, we put $\beta(v) := \mathrm{N}^+_D(v) \in G$ for all $v \in V$; Given β, we construct the digraph D by setting $\mathrm{A}_D := \{vw : w \in \beta(v)\}$. Indeed, both the map β and the digraph D can be further represented as the $V \times V$ $(0, 1)$-matrix whose v, w-entry is 1 if and only if $w \in \beta(v)$, hence also standing for a linear map from the binary linear space 2^V to itself. We often freely switch among these few equivalent objects and this viewpoint explains how the binary linear algebra could come into play.

In his Presidential Address (General) to the Indian Mathematical Society Annual Conference held at University of Pune, December 2007, Ravindra Bapat [4] made the following claim:

> We are made to believe that physics, chemistry or biology are easier to explain to the general audience. $\cdots$ Their abstract concepts are presented in a way as if they are actual realities. Atoms, quarks, dark energy, strings, black holes, are all concepts but people believe in them. In comparison mathematicians are awfully shy of presenting anything for which they do not have a refereed proof. We need to be bold.

The concept of a group first appeared in the modern sense in the mind of Évariste Galois. Encouraged by the above quotation from Bapat, let us be brave enough to talk about the idea of a group without giving any definition of it here. If necessary, the reader could go to [2, 42] to check relevant definitions and see how groups measure the amount of symmetry.

Taking a group G, a set S and a map $\beta \in G^S$, the *Cayley digraph* $\Gamma(G, \beta)$ has vertex set G and arc set $\{(g, g\beta(s)) \colon s \in S, g \in G\}$. The structure of many families of Cayley digraphs and some related mathematical objects have been widely investigated in various guises in several mathematical fields [1, 29, 36, 38]. Assume that we are travelling in a world modelled by the Cayley digraph $\Gamma(G, \beta)$. Each element $s \in S$ is a vehicle which can take us from where we are, say $g \in G$, to $g\beta(s) \in G$. The geometry of $\Gamma(G, \beta)$ is what we should concern with when we know that every vehicle $s \in S$ is available everywhere. But what if each vehicle $s \in S$ is available only at a specific set of stations? To be more precise, we now have a map α from S to 2^G, which specifies the "lit-only restriction", and the real world which we live in becomes the *lit-only Cayley digraph* $\Gamma(G, \beta, \alpha)$ that has vertex set G and arc set $\{(g, g\beta(s)) \colon s \in S, g \in \alpha(s)\}$. If $\alpha(s) = G$ for all $s \in S$, surely $\Gamma(G, \beta, \alpha)$ is just $\Gamma(G, \beta)$.

Fix a set V. The power set of V, namely 2^V, forms a vector space over $\mathbb{F}_2$ where the sum $x + y$ of two elements x and y of 2^V is given by the symmetric difference $x \triangle y$ of them. The empty set $\emptyset$ is the zero element in 2^V, and it is usually denoted by $\mathbf{0}$ to emphasize that it is an element of the vector space 2^V. Take any map β from V to 2^V and let α be the map from V to 2^V such that $\alpha(v) := \{x \in 2^V \colon v \in x\}$. Recall that the map β corresponds to a digraph D with vertex set V. Playing the *σ-game* on the digraph D is just to start from any vertex of $\Gamma(2^V, \beta)$, that is, a subset of V, and to find a "good" path in $\Gamma(2^V, \beta)$ which ends at some "good" vertex. Here, a good path may refer to a path of minimum length and a good vertex may refer to a subset of V of minimum possible size. Similarly, playing the *lit-only σ-game* on the digraph D is just to start from any vertex of $\Gamma(2^V, \beta, \alpha)$ and to find a "good" path in $\Gamma(2^V, \beta, \alpha)$ which ends at some "good" vertex. We often write $\mathcal{PS}_D$ and $\mathcal{PS}^{\sigma}{}_D$ for $\Gamma(2^V, \beta, \alpha)$ and $\Gamma(2^V, \beta)$, respectively, and call them the *phase space of the lit-only σ-game* and the *phase space of the σ-game* on the digraph D, respectively. In the combinatorial games community, people use the concept of "game graph" [22] with the same meaning of our "phase space" here. We point the readers to [26, Appendix A] for the drawings of some concrete phase spaces of the lit-only σ-game. For $x, y \in 2^V$, we write $x \to_{\sigma D} y$ to mean that there is a path from x to y in $\mathcal{PS}^{\sigma}{}_D$ and we use $x \to_D y$ to indicate that there is a path in $\mathcal{PS}_D$ leading from x to y. Note that $x \to_{\sigma D} y$ follows from $x \to_D y$.

The process shown in Fig. 1.2 can be recorded as $\{a,b,c,d,e\} \xrightarrow{e} \{a,d\} \xrightarrow{b}_{\sigma} \{b,c,d,e\} \xrightarrow{c} \mathbf{0}$. We may even write directly $\{a,b,c,d,e\} \xrightarrow{ebc}_{\sigma} \mathbf{0}$ or $\{a,b,c,d,e\} \to_{\sigma} \mathbf{0}$. Similarly, the process shown in Fig. 1.3 can be represented as $\{a,b,c,d,e\} \xrightarrow{bdced} \mathbf{0}$ or $\{a,b,c,d,e\} \to \mathbf{0}$.

Among many research problems on the lit-only σ-game, let us mention a natural one here to which every lit-only σ-game player must want to know the answer.

Problem 2.1 (The lit-only σ-game reachability problem)**.** Let D be a digraph and we have as input two subsets x and y of V_D. Can we decide efficiently whether or not $x \to_D y$?

Since the size of the phase space $\mathcal{PS}_D$ is exponential in $|\,\mathrm{V}_D\,|$, it may even be unclear if the above general decision problem is in NP. Let us invite the reader to tackle some special easy cases: What about $x = \mathbf{0}$ and $y \neq \mathbf{0}$? Or, $x \neq \mathrm{L}_D$ and $y = \mathrm{L}_D$?

Let D be a graph. A classic result, called Sutner's Theorem and proved repeatedly in the literature before and after Sutner [3, 44, 45], asserts that

$$\mathrm{L}_D \to_{\sigma\, D} \mathbf{0}\,. \tag{2.1}$$

Goldwasser and Klostermeyer [24] wondered whether or not it indeed holds

$$\mathrm{L}_D \to_D \mathbf{0} \tag{2.2}$$

on the condition that $\mathrm{L}_D = \mathrm{V}_D$. After knowing this problem from a talk of John Goldwasser at Shanghai Jiao Tong University (SJTU) in the spring of 2008, Mr. Mu Li, an undergraduate student of SJTU then, was able to work out a short proof for a positive answer in a few days. However, after some search in the internet, Mu Li found the Lights Out webpage of Scherphuis and noticed that the same result had been announced together with a proof earlier there. Note that Scherphuis asserted on his webpage that several of his attempts at a proof had holes and it is remarkably hard to really get a proof. Inspired by the discovery of Li, Goldwasser and his two new friends in China, Xinmao Wang and Yaokun Wu, then turned to the project of understanding how much difference the lit-only restriction will make for playing the σ-game. An outcome of Goldwasser's journey to China in 2008 spring is the paper [27] in which you find the following generalization of the result of Scherphuis: Assuming that D is a connected graph with $\mathrm{V}_D = \mathrm{L}_D$ and that $x, y \in 2^{\mathrm{V}_D}$ are two configurations satisfying $x \neq \mathbf{0}$ and $y \neq \mathrm{V}_D$, then $x \to_{\sigma\, D} y$ if and only if $x \to_D y$. Finally, let us mention that Eq. (2.2) indeed holds for all graphs D and a proof of it comes from a simple application of the Gaussian elimination [52]. This fact just explains what you saw from Figs. 1.2 and 1.3. We hope that Sutner's Theorem (Eq. (2.1)) and its lit-only version (Eq. (2.2))

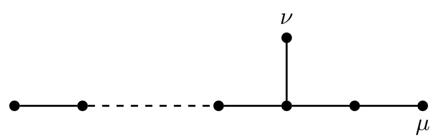


Figure 2.1. The ADE graphs.

will enable you challenge and surprise elementary school students by leaving them many different digraphs, which should have a nonempty set of loops, of course.

This note will focus more on the lit-only version of the σ-game and so let us prepare one more tiffin (light meal) about it. A *tree* is a connected graph without cycles. Let E_n be the n-vertex tree as shown in Fig. 2.1. Let $D_{n-1} = E_n - \mu$ and let $A_{n-2} := D_{n-1} - \nu$. Note that A_n, D_n, E_6, E_7, E_8 are known as simply-laced Dynkin diagrams. In Fig. 2.2, we show an example of decreasing the size of a set from two to one in the lit-only σ-game on E_6. What is behind Fig. 2.2 is the Borel-De Siebenthal Theorem [8], which says that, whichever you start in the lit-only σ-game on a simply-laced Dynkin diagram, you always have a strategy to reach a configuration of size at most one. Going one step further, Hsin [31, Corollary 3.2] finds that a connected graph is strictly Borel-De Siebenthal if and only if it is one of the ADE graphs defined above, where a graph is called *strictly Borel-De Siebenthal* if, for the lit-only σ-game on any of its connected subgraphs, every given configuration can evolve into a one of size at most one.

A brief history of the σ-game is reported in [44]. There are vast researches on the σ-game, say [5, 19, 25, 54, 55], from which you can find out many interesting stuff for children. Due to the invention of Vogan diagram [37] in the study of Lie algebra, some Lie theorists have been good players of the lit-only σ-game [6, 11, 12, 18, 23, 32, 33, 41, 43]. The lit-only σ-game and its close variants also appear in other interesting contexts; see [9, 16, 17, 20, 21, 34, 35].

At this point, we recommend to stop reading this note, click on the links given above or in our references and continue there or simply turn to the fundamental problem for playing the lit-only σ-game, Problem 2.1. It is normal to view Problem 2.1 as an algorithmic problem; but an algorithm is nothing but an evolution rule of a dynamics or a design of a Markov chain. So, if your parents want you prepare for your university life, say to read calculus for the study of various dynamical systems, you can still have fun with Problem 2.1 by claiming that you are working hard on a calculus-free dynamical system. In case you insist and agree with Bertrand Russell that ordinary language is totally unsuited for expressing what physics really asserts, please be ready to follow a sequel of (strange!) definitions below, which we arrive at after playing the lit-only σ-game for some while, and then, hopefully, share our excitement and curiosity on what those concepts

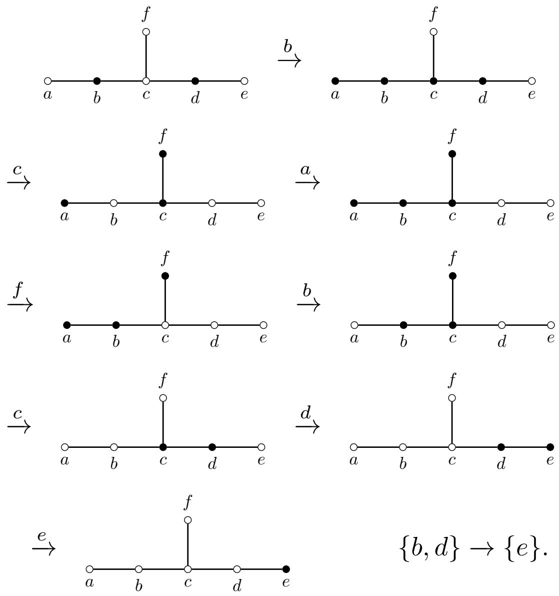


Figure 2.2. Playing the lit-only σ-game on E_6.

will bring forth. Note that after providing more ideas on our tour in the land of the lit-only σ-game, in § 6 we will be able to sketch our partial solution to the algorithmic problem mentioned above.

3. A strange goose

We increase our stock of observations on the lit-only σ-game in this section, demonstrating that it is really a strange goose. We are mainly interested in the phase spaces of the game and the possible difference caused by the lit-only restriction.

3.1. **Minimum light number.** The puzzles which you can bring back home for kids from § 2 are all about how many lights can be switched off, namely the minimum light number. Instead of developing tricks on solving those puzzles faster, let us show something about the influence of the lit-only restriction from a mathematical perspective.

Take a digraph D. For any $x \in 2^{\mathrm{V}_D}$, define $\mathsf{ML}_D(x) := \min\{|y| :\ x \to_D y\}$ and $\mathsf{ML}^\sigma{}_D(x) := \min\{|y| :\ x \to_{\sigma D} y\}$. Let

$$\mathsf{ML}(D) := \max_{x \in 2^{\mathrm{V}_D}} \mathsf{ML}_D(x) \text{ and } \mathsf{ML}^\sigma(D) := \max_{x \in 2^{\mathrm{V}_D}} \mathsf{ML}^\sigma{}_D(x).$$

For any graph G and $v \in \mathrm{V}_G$, let $\deg_G(v) := |\{e \in \mathrm{E}_G :\ v \in e\}|$. A *leaf* of a tree G is a vertex v with $\deg_G(v) = 1$.

On August 17, 2005, Gerard Jennhwa Chang gave a plenary talk on Lie algebra in the Third Pacific Rim Conference on Mathematics held in Fudan University, Shanghai. Yaokun Wu tried to run across Shanghai to sit in this talk but was only able to catch the last one third of it. In the summer of 2006, Wu attended a three-week summer school on topology in Guizhou University and he met his former classmate Xinmao Wang there. They decided to do something together for fun in their spare time during the summer school. What they obtained in that cool summer in Guiyang is the following, which was inspired by a corresponding conjecture posed in the talk of Chang.

Theorem 3.1 ([47, 48])**.** *Let T be a tree with $\ell > 1$ leaves. Then* $\mathsf{ML}(T) \leq \lceil \frac{\ell}{2} \rceil$ *and* $\mathsf{ML}^\sigma(T) \leq \lfloor \frac{\ell}{2} \rfloor$.

Conjecture 3.2 ([26, Conjecture 14] [49, Conjecture 4, Conjecture 7])**.** It holds $\mathsf{ML}(G) - \mathsf{ML}^\sigma(G) \leq \frac{|\mathrm{V}_G|}{2}$ for any graph G. If G is a tree with zero or more loops attached, then $\mathsf{ML}(G) - \mathsf{ML}^\sigma(G) \leq 1$. If G is obtained from a unicyclic graph by adding zero or more loops, then $\mathsf{ML}(G) - \mathsf{ML}^\sigma(G) \leq 2$.

Conjecture 3.3 ([26, Conjecture 5] [49, Conjecture 15])**.** Let G be a connected graph on n vertices. Then $\max_{x \in 2^{\mathrm{V}_G}} (\mathsf{ML}_G(x) - \mathsf{ML}^\sigma{}_G(x)) \leq \max_{v \in \mathrm{V}_G} \deg_G(v) - 1$. Moreover, if $\max_{x \in 2^{\mathrm{V}_G}} (\mathsf{ML}_G(x) - \mathsf{ML}^\sigma{}_G(x)) > \frac{n}{2}$, then G is a complete m-partite graph for some positive integer $m \equiv 3 \pmod 4$.

3.2. **Reachability of the phase spaces.** Let D be a digraph and let S be the set of its strongly connected components. Let R be the set of ordered pairs (S_1, S_2) where S_1 and S_2 are different elements of S and there exists in D a path leading from a vertex in S_1 to a vertex in S_2. It is clear that R is a transitive acyclic relation on S, namely (S, R) is a poset, and this relation gives exactly the reachability relations in D. The *condensation* of the digraph D, denoted by $\mathrm{CD}(D)$, is the Hasse diagram of the poset (S, R).

The condensation operation helps us obtain a more readable picture of the phase spaces when we are only concerned with the reachability issue. Note that $\mathrm{CD}(\mathcal{PS}^\sigma{}_D)$ consists of isolated vertices as $\mathcal{PS}^\sigma{}_D$ is symmetric. Moreover, for vertices $x, y \in 2^{\mathrm{V}_D}$, $x \to_D y$ if and only if there exists a directed path from strongly connected component containing x to strongly connected component containing y in $\mathrm{CD}(\mathcal{PS}_D)$. Therefore, given $x, y \in 2^{\mathrm{V}_D}$, to test $x \to_D y$ ($x \to_{\sigma D} y$) we can work in two steps, to determine $\mathrm{CD}(\mathcal{PS}_D)$ ($\mathrm{CD}(\mathcal{PS}^\sigma{}_D)$) and then to determine in which strongly components of $\mathcal{PS}_D$ (components of $\mathcal{PS}^\sigma{}_D$) are x and y.

For any digraph D, $\{x \in 2^{\mathrm{V}_D} : \mathbf{0} \to_{\sigma D} x\}$ is a vector subspace of 2^{V_D}, which we call the *neighbour space* of D and denote by $\mathcal{N}_D$. We will always use rank_D to designate the dimension of $\mathcal{N}_D$ and use corank_D for $|\mathrm{V}_D| - \mathrm{rank}_D$, and call them the *rank* and *corank* of D, respectively. Note that the rank of a symmetric loopless digraph is always an even integer. We call a coset $\mathcal{M}$ of $\mathcal{N}_D$ in 2^{V_D} which is not $\mathcal{N}_D$ itself a *shifted neighbour space* of D. The concept of a neighbour space seems to be of little appearance in algebraic graph theory; in contrast, cut space and cycle space of a graph are much more popular there. However, in coding theory, neighbour spaces are well studied in the context of linear codes [14], or just binary codes if the coefficients are from the binary field as here. In the study of error-correcting code, $\mathsf{ML}^\sigma(D)$ is known as the covering radius of the binary code $\mathcal{N}_D$ [30]. The neighbour space $\mathcal{N}_D$ is of interest to the σ-game players as its cosets $\mathcal{M}$ are exactly the set of all strongly connected components of the symmetric digraph $\mathcal{PS}^\sigma{}_D$, that is, the reachability relation $\rightarrow_{\sigma}{}_D$ coincides with $\bigcup_{\mathcal{M}} \mathcal{M} \times \mathcal{M}$ where $\mathcal{M}$ runs through all cosets of $\mathcal{N}_D$. In other words, $\mathrm{CD}(\mathcal{PS}^\sigma{}_D)$ is the empty digraph with 2^{corank_D} vertices, each of which is a coset of $\mathcal{N}_D$.

It is clear that $\mathcal{PS}_D$ is a subgraph of $\mathcal{PS}^\sigma{}_D$ and so $x \rightarrow_D y$ could happen only when x and y come from the same coset of $\mathcal{N}_D$. So, $\mathrm{CD}(\mathcal{PS}_D)$ is the disjoint union of $\mathrm{CD}(\mathcal{PS}_D[\mathcal{M}])$ where $\mathcal{M}$ runs over all cosets of $\mathcal{N}_D$. If you have solved the easy questions mentioned after Problem 2.1, you will know that $\mathbf{0}$ and L_D are really interesting configurations for the lit-only σ-game. Indeed, $\mathbf{0}$ is a sink in $\mathcal{PS}_D$ and L_D is a source in $\mathcal{PS}_D$, and hence both $\{\mathbf{0}\}$ and $\{\mathrm{L}_D\}$, which are not necessarily distinct, form strongly connected components in $\mathcal{PS}_D$. Recall from Eq. (2.1) that $\{\mathbf{0}, \mathrm{L}_D\} \subseteq \mathcal{N}_D$. Surprising as it may first seem, essentially, this gives almost all the difference between the two reachability relationships $\rightarrow_{\sigma}{}_D$ and $\rightarrow_D$. We clarify a bit what does it mean in the rest of this section.

To really see all possible differences caused by the lit-only restriction, we invited the computer to play the game. We say that a digraph is *loop-linked* provided every vertex can reach and be reached by a loop vertex in it. Our first try resulted in the discovery of [26, Conjecture 6], which says that, when D is a loop-linked strongly connected symmetric digraph, the differences between $\rightarrow_{\sigma}{}_D$ and $\rightarrow_D$ can behave in two ways. When the first-named author introduced the lit-only σ-game in a course for freshmen in the winter of 2010, the second-named author was among those freshmen and thus the game was played a lot by him and his laptop then. Going beyond [26, Conjecture 6], our computer experiment suggests a classification of all strongly connected digraphs D into altogether fourteen classes, eight of them being graphs and six of them being asymmetric digraphs, such that for digraphs D from the same class the difference between $\rightarrow_{\sigma}{}_D$ and $\rightarrow_D$ can be said to be the same. For all possible fourteen cases at most one shifted neighbour space of D is not strongly connected – we will call that possible only shifted neighbour space the *exceptional shifted neighbour space*. Moreover, $\mathrm{CD}(\mathcal{PS}_D(\mathcal{M}))$ is either an empty digraph or a directed path for exceptional shifted neighbour space $\mathcal{M}$.

A deep learning from the data generated by our computer experiment finally yields a proof of our guess for all connected graphs as well as a recognition of those eight classes of connected graphs. This means that we could have a paper-and-pencil proof for what we see for the lit-only σ-game on connected graphs and we can have a definition/characterization for those eight graph classes independent of the dynamical behaviors of the lit-only σ-game on them. These eight graph classes include two classes of loop-linked graphs and six classes of loopless graphs. Due to our structural characterization of the eight graph classes, the two loop-linked graph classes have been named loop-linked line graphs and loop-linked cuspidal graphs while the six loopless graph classes have been named opal line graphs, polished emerald line graphs, unpolished emerald line graphs, polished cuspidal graphs of type I, polished cuspidal graphs of type II, and unpolished cuspidal graphs [52]. We leave the afore-mentioned structural definition of these eight graph classes to Definitions 4.1 and 4.7. In Table 3.1, we depict $\mathrm{CD}(\mathcal{PS}_D)$ for any connected graph D according to which of the eight classes it belongs to. Indeed, the line labelled by "neighbour space" shows the digraph $\mathrm{CD}(\mathcal{N}_D)$; if there exists an exceptional shifted neighbour space $\mathcal{M}$ we will add a line labelled by "exception" to show the condensation digraph $\mathrm{CD}(\mathcal{M}_D)$ of that exceptional shifted neighbour space. Note that the number in each ellipse refers to the size of the corresponding strongly connected component represented by that ellipse. Also note that the two singleton sets $\{\mathbf{0}\}$ and $\{\mathrm{L}_D\}$ always appear as vertices of $\mathrm{CD}(\mathcal{N}_D)$ and we simply use bullets to represent them. In Table 3.1, we do not bother to draw anything about a shifted neighbour space that is not exceptional as we know that it must be a strongly connected component of both $\mathcal{PS}_D$ and $\mathcal{PS}^{\sigma}{}_D$ of size $2^{\operatorname{rank}\mathcal{N}_D}$. With this convention, we can list a main result from [52].

Theorem 3.4 ([52])**.** *The set of all connected graphs D can be classified into eight types according to* $\mathrm{CD}(\mathcal{PS}_D)$ *as shown in Table 3.1.*

By Theorem 3.4, for any connected loop-linked graph D, $x \to_D \mathbf{0}$ is equivalent to $x \to_{\sigma D} \mathbf{0}$. According to Eq. (2.1), this fact can be said to be a generalization of Eq. (2.2). Although Eq. (2.2) possesses a short proof, we are still unaware of any simple proof of this more general result.

What we saw from computer experiment convinces us that strongly connected asymmetric digraphs can be classified into six classes, five of them being loop-linked and one of them loopless. This is summarized briefly below, where Table 3.2 should be read with the convention of reading Table 3.1.

Conjecture 3.5 ([52])**.** Strongly connected asymmetric digraphs D can be classified into six classes as shown in Table 3.2 according to $\mathrm{CD}(\mathcal{PS}_D)$.

Graph class	Phase space	
Six classes of loopless graphs:		
1. Opal line graphs	Neighbour space	$\mathbf{0}\bullet \quad \binom{r+1}{2} \quad \binom{r+1}{4} \quad \cdots \quad \binom{r+1}{r-2} \quad \binom{r+1}{r}$
2. Polished emerald line graphs	Neighbour space	$\mathbf{0}\bullet \quad \binom{r+2}{2} \quad \binom{r+2}{4} \quad \cdots \quad \binom{r+2}{r/2-1} \quad \binom{r+2}{r/2+1}$
	Exception	$\binom{r+2}{1} \quad \binom{r+2}{3} \quad \cdots \quad \binom{r+2}{r/2-2} \quad \binom{r+2}{r/2}$
3. Unpolished emerald line graphs	Neighbour space	$\mathbf{0}\bullet \quad \binom{r+2}{2} \quad \binom{r+2}{4} \quad \cdots \quad \binom{r+2}{r/2-2} \quad \binom{r+2}{r/2}$
	Exception	$\binom{r+2}{1} \quad \binom{r+2}{3} \quad \cdots \quad \binom{r+2}{r/2-1} \quad \binom{r+2}{r/2+1}$
4. Polished cuspidal graphs, I	Neighbour space	$\mathbf{0}\bullet \quad 2^{r-1}-2^{r/2-1} \quad 2^{r-1}+2^{r/2-1}-1$
5. Polished cuspidal graphs, II	Neighbour space	$\mathbf{0}\bullet \quad 2^{r-1}-2^{r/2-1}-1 \quad 2^{r-1}+2^{r/2-1}$
6. Unpolished cuspidal graphs	Neighbour space	$\mathbf{0}\bullet \quad 2^{r}-1$
	Exception	$2^{r-1}-2^{r/2-1} \quad 2^{r-1}+2^{r/2-1}$
Two classes of loop-linked graphs:		
7. Loop-linked line graphs	Neighbour space	$\mathbf{0}\bullet\!\leftarrow \binom{r}{1} \leftarrow \binom{r}{2} \leftarrow \cdots \leftarrow \binom{r}{r-2} \leftarrow \binom{r}{r-1} \leftarrow\!\bullet\ \mathrm{L}_D$
8. Loop-linked cuspidal graphs	Neighbour space	$\mathbf{0}\bullet\!\leftarrow 2^{r}-2 \leftarrow\!\bullet\ \mathrm{L}_D$

Table 3.1. The classification of all connected graphs D according to $\mathrm{CD}(\mathcal{PS}_D)$; see Theorem 3.4.

Graph class	Phase space	
9. Loopless asymmetric digraphs	Neighbour space	$\mathbf{0}\ \bullet \qquad 2^r-1$
Five classes of loop-linked asymmetric digraphs:		
10. Asymmetric digraphs, I	Neighbour space	$\mathbf{0}\ \bullet\!\leftarrow \binom{r}{1} \leftarrow \binom{r}{2} \leftarrow \cdots \leftarrow \binom{r}{r-2} \leftarrow \binom{r}{r-1} \leftarrow\!\bullet\ \mathrm{L}_D$
11. Asymmetric digraphs, IIA	Neighbour space	$\mathbf{0}\ \bullet\!\leftarrow \binom{r+1}{2} \leftarrow \binom{r+1}{4} \leftarrow \cdots \leftarrow \binom{r+1}{r-2} \leftarrow \binom{r+1}{r}$
	Exception	$\binom{r+1}{1} \leftarrow \binom{r+1}{3} \leftarrow \cdots \leftarrow \binom{r+1}{r-3} \leftarrow \binom{r+1}{r-1} \leftarrow\!\bullet\ \mathrm{L}_D$
12. Asymmetric digraphs, IIB	Neighbour space	$\mathbf{0}\ \bullet\!\leftarrow \binom{r+1}{2} \leftarrow \binom{r+1}{4} \leftarrow \cdots \leftarrow \binom{r+1}{r-3} \leftarrow \binom{r+1}{r-1} \leftarrow\!\bullet\ \mathrm{L}_D$
	Exception	$\binom{r+1}{1} \leftarrow \binom{r+1}{3} \leftarrow \cdots \leftarrow \binom{r+1}{r-2} \leftarrow \binom{r+1}{r}$
13. Asymmetric digraphs, IIIA	Neighbour space	$\mathbf{0}\ \bullet\!\leftarrow 2^r-2 \leftarrow\!\bullet\ \mathrm{L}_D$
14. Asymmetric digraphs, IIIB	Neighbour space	$\mathbf{0}\ \bullet\!\leftarrow 2^r-1$
	Exception	$2^r-1 \leftarrow\!\bullet\ \mathrm{L}_D$

Table 3.2. A possible classification of all strongly connected asymmetric digraphs; see Conjecture 3.5.

We fail to verify Conjecture 3.5 so far. But, according to Avul Pakir Jainulabdeen Abdul Kalam, an aerospace scientist and the 11th President of India, F.A.I.L. only means "First Attempt In Learning". Our first attempt does demonstrate some small theoretical evidence for the truth of Conjecture 3.5. Let D be a strongly connected digraph. For every vertex $v \in \mathrm{V}_D$, let ϕ_v denote the set of vertices w of D for which there is a path in D from v to w such that $v \notin \mathrm{N}_D^+(u)$ for all vertices u in the path. Under the assumption that D contains a pair of its vertices v and w such that $\phi_v \cup \phi_w = \mathrm{V}_D$, we can verify Conjecture 3.5. Indeed, the phase space of such a digraph could only take the form as described in cases 6 and 8 in Table 3.1 or cases 9, 13 and 14 in Table 3.2 [52]. For a possible confirmation of Conjecture 3.5, it may be important to find, for each of the possible five classes of loop-linked asymmetric digraphs as indicated in Table 3.2, a structural characterization stated irrelevant with any specific dynamical process on the digraphs.

4. Some mathematical eggs

Let us exhibit some mathematical eggs laid by our strange goose and collected in [52]. We acknowledge that the story went deeper than we had imagined and so we do feel like the man in Fig. 4.1: are we getting paid for eating ice-cream? Here is a grook and a superegg from Piet Hein [4], which do suggest us taking fun in earnest:

The Eternal Twins

Taking fun
as simply fun
and earnestness
in earnest
shows how thoroughly
thou none
of the two
discernest.

A superegg lamp.
https://www.piethein.com/product/superegg-300-1901/

4.1. **Line graphs and cuspidal graphs.**

During his visit to Shanghai in 2008 spring, John Goldwasser suggested to analyze the shape of the phase space of the lit-only σ-game on (symmetric) paths. Following this suggestion, Mu Li discovered nice inductive patterns from computer experiments on paths and then

[4]He often plays with blocks and sticks. See: https://www.piethein.com/category/games-36/.

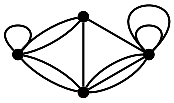
Figure 4.1. The goose that laid the golden eggs.
By Milo Winter (1888–1956).

https://www.ancient.eu/image/10121/the-goose-that-laid-the-golden-egg/

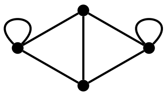
he was working with Yaokun Wu towards a paper on the very explicit description of $\mathcal{PS}_D$ for all paths D. The manuscript was not polished to a stage to be submitted anywhere, unfortunately. But, the pleasant applications of inductive arguments/descriptions in this project did direct Wu to think about the underlying geometry which controls the dynamics and then he found a connection between the lit-only σ-game and the concept of "line graphs" in [51]. We point out that the line graph concept used in [51] is the classical one as introduced by Hassler Whitney [50] and does not coincide with our usage of line graphs in Table 3.1. Indeed, those "line graphs" as defined by Whitney correspond to our loopless ordinary line graphs in view of Definition 4.1 to be presented below.

The phase space structure as shown on the right of the last two lines of Table 3.1 has been predicted in [26, Conjecture 6]. The first significant step for us to prove that conjecture is to have a correct guess of the geometric structure which causes those two different dynamical behaviours. By trial-and-error, we divide the class of all graphs into the class of line graphs and the class of cuspidal graphs. To show you what these two graph classes are, it suffices to define line graphs.

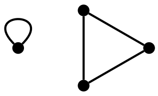
A *multigraph* G consists of a nonempty vertex set V_G, an edge set E_G, and a map ∂_G from E_G to $\binom{\mathrm{V}_G}{1} \cup \binom{\mathrm{V}_G}{2}$, called the *boundary map* of G. If $\mathrm{E}_G = \emptyset$, G is said to be an *empty* (multi)graph. When the boundary map is injective, we can identify an edge with its boundary, namely its image under the boundary map, and thus the multigraph in consideration becomes a graph (symmetric digraph). You may think of a multigraph as a symmetric nonnegative integer matrix while a graph as a symmetric $(0,1)$-matrix. There are two natural maps $\mathbb{N} \to \{0,1\}$: Viewing $\{0,1\}$ as the Boolean semifield $\mathbb{B}$, we send 0 to 0 and all positive integers to 1; Viewing $\{0,1\}$ as the binary field $\mathbb{F}_2$, we send every nonnegative integer n to $n \bmod 2$. Accordingly, as multigraphs are just nonnegative integer symmetric matrices, we have two natural maps from multigraphs to graphs. For example, the multigraph in Fig. 4.2(a) gives rise to two graphs in Fig. 4.2 4.2(b) and 4.2(c), depending on which viewpoint we will adopt. In the literature, most definitions on line graphs and various variants use the Boolean semifield approach. We will go the other way here.



(a) A multigraph G.

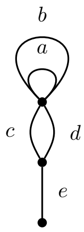


(b) G over $\mathbb{B}$.

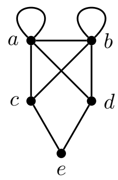


(c) G over $\mathbb{F}_2$.

Figure 4.2. Two ways to view a multigraph as a graph.



(a) G



(b) $\mathfrak{L}(G)$

Figure 4.3. A multigraph and its line graph.

Definition 4.1. Given a nonempty multigraph G, its *line graph*, denoted by $\mathfrak{L}(G)$, is the graph with vertex set $\mathrm{V}_{\mathfrak{L}(G)} := \mathrm{E}_G$ and edge set

$$\mathrm{E}_{\mathfrak{L}(G)} := \{\{e,f\} : |\partial_G(e) \cap \partial_G(f)| \equiv 1 \pmod 2,\ e,f \in \mathrm{E}_G\}. \tag{4.1}$$

If G has no isolated vertices, we will call G a *root multigraph* of $\mathfrak{L}(G)$. A line graph is *ordinary* if it is the line graph of a graph.

Theorem 4.2 ([7]). *A loopless graph is an ordinary line graph if and only if it does not contain any of the nine graphs from Fig. 4.4 as a vertex-induced subgraph.*

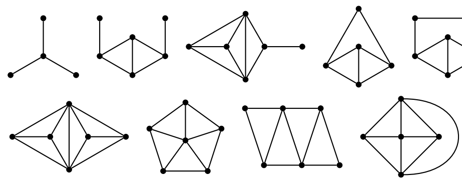


Figure 4.4. Nine forbidden vertex-induced subgraphs for loopless ordinary line graphs.

For an ordinary loopless line graph $\mathfrak{L}(G)$ and any $v \in \mathrm{V}_{\mathfrak{L}(G)}$, if we add several new vertices to $\mathfrak{L}(G)$ and make them adjacent with every vertex from $\mathrm{N}_{\mathfrak{L}(G)}(v)$, we will obtain a new loopless line graph with a root multigraph that can be obtained from G by adding several edges parallel to v. Note that, root multigraphs may not be unique. In Fig. 4.5, we show three forbidden vertex-induced subgraphs for loopless line graphs in Fig. 4.4 are actually line graphs. Moreover, for each of them a root multigraph is presented. In Fig. 4.5, one can also see how those parallel edges in the root multigraphs correspond to the independent set in the line graph which share the same neighbours.

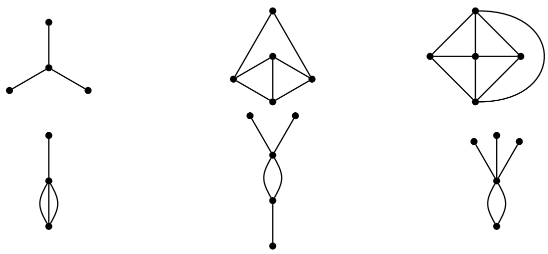


Figure 4.5. Three line graphs from the list of Beineke (Theorem 4.2) and their root multigraphs.

Given a loopless graph G, to construct its line graph $\mathfrak{L}(G)$, which is ordinary by definition, Eq. (4.1) can be rewritten as

$$\mathrm{E}_{\mathfrak{L}(G)} := \{\{e,f\} : \ |\partial_G(e) \cap \partial_G(f)| > 0, \{e,f\} \in \binom{\mathrm{E}_G}{2}\}. \tag{4.2}$$

Comparing Eqs. (4.1) and (4.2), what we introduced in [52] as line graphs can be said to be a "mod two" version of the usual line graphs. Our exposure to the lit-only σ-game enables us [52] take the "mod two" definition and that has made all the difference.

A graph is called *nonsingular* if its associated matrix is nonsingular over the binary field. In all, there are 43 connected nonsingular 6-vertex graphs. They comprise of the 32 forbidden graphs from Fig. 4.6 and the 11 line graphs of 7-vertex trees from Fig. 4.7.

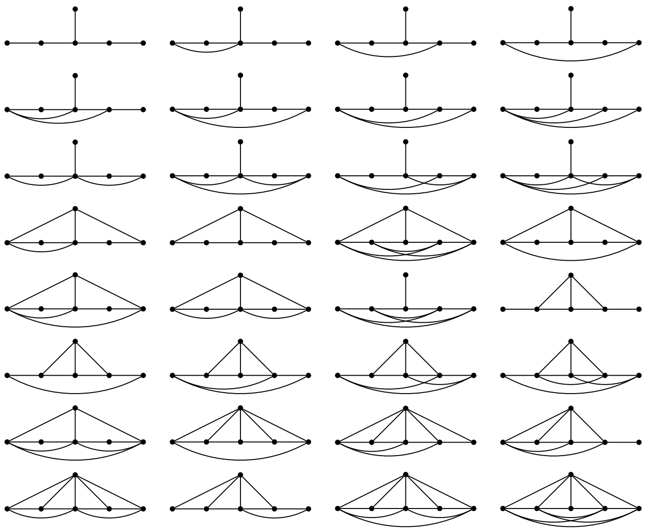


Figure 4.6. The 32 forbidden subgraphs for loopless line graphs.

Theorem 4.3 ([52])**.** *For a loopless graph G, the following statements are equivalent.*

- *The graph G is a line graph.*
- *The graph G does not contain any graph from the thirty-two graphs depicted in Fig. 4.6 as an induced subgraph.*
- *Every connected 6-vertex vertex-induced subgraph of G is a line graph.*
- *Every connected nonsingular 6-vertex vertex-induced subgraph of G is one of the eleven line graphs of 7-vertex trees as shown in Fig. 4.7.*

Let G be a graph and v be a loop vertex of G. Let $G \boxminus v$ be the graph with vertex set $\mathrm{V}_G \setminus \{v\}$ and edge set $\mathrm{E}_G \triangle \binom{\mathrm{N}_G(v)}{1} \triangle \binom{\mathrm{N}_G(v)}{2}$. We say that $G \boxminus v$ is obtained from G by *melting a loop* v.

Theorem 4.4 ([52])**.** *A loop-linked graph is cuspidal if and only if it can be reduced to one of the two graphs in Fig. 4.9 by a sequence of deleting vertex operation and melting loop operation through loop-linked graphs.*

William Thurston made the following claim:

> The product of mathematics is clarity and understanding. Not theorems, by themselves.

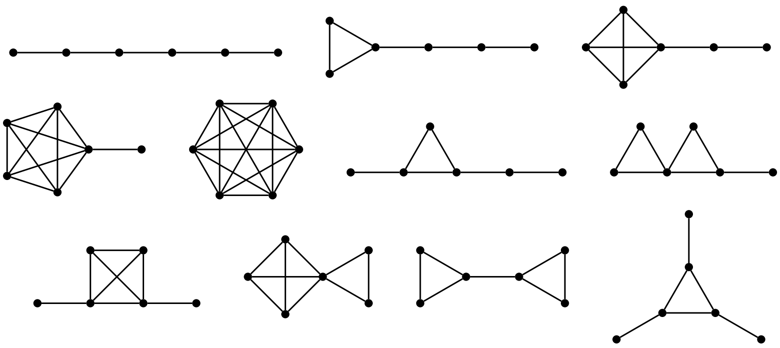


Figure 4.7. Eleven 6-vertex line graphs of trees.

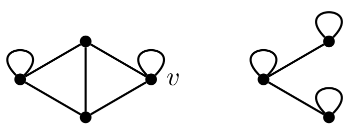


Figure 4.8. A graph G with a loop v, and $G \boxminus v$.

Figure 4.9. The two minimal (with respect to operators $-$ and $\boxminus$) loop-linked cuspidal graphs.

Will Theorems 4.3 and 4.4 and other results about line graphs in [52] stimulate you to do some mathematics and yield a better understanding of some nice new structure?

4.2. **Quardratic forms and bilinear forms.** Let D be a digraph. For each $v \in \mathrm{V}_D$, let v^* be the linear functional on 2^{V_D} such that, for all $x \in 2^{\mathrm{V}_D}$,

$$v^*(x) :\equiv \begin{cases} 1, & v \in x; \\ 0, & v \notin x. \end{cases}$$

For every $v \in \mathrm{V}_D$, construct a map $\mathcal{T}_{D,v} \in \mathsf{End}(2^{\mathrm{V}_D})$:

$$x \mapsto \begin{cases} x \triangle \mathrm{N}_D^+(v), & v \in x, \\ x, & v \notin x. \end{cases}$$

Note that $\mathcal{T}_{D,v} = \mathrm{Id}_{2^{\mathrm{V}_D}} + \mathrm{N}_D^+(v)v^*$ and hence a transvection. Clearly, $\mathcal{T}_{D,v}$ is invertible if and only if $v \notin \mathrm{L}_D$. The *lit-only monoid of D* is the multiplicative monoid generated by $\mathcal{T}_{D,v}$'s:

$$\mathsf{LOM}_D := \langle \mathcal{T}_{D,v} :\ v \in \mathrm{V}_D \rangle.$$

To study the lit-only σ-game on D is to understand the action of LOM_D on 2^{V_D}. If D is loopless, the lit-only monoid of D is indeed a subgroup of $\mathsf{SL}(2^{\mathrm{V}_D})$ and we call it the *lit-only group* of D; in this case, we often write LOG_D for LOM_D accordingly. Since $\mathcal{N}_D$ is an invariant subspace under the action of LOG_D, there is a natural restriction homomorphism $\cdot|_{\mathcal{N}_D} :\ \mathsf{LOG}_D \to \mathsf{SL}(\mathcal{N}_D)$.

Definition 4.5. Let D be a loopless digraph. Its *topaz group*, denoted by TG_D, is the image of the natural restriction homomorphism $\cdot|_{\mathcal{N}_D}$, namely

$$\mathsf{TG}_D := \langle g|_{\mathcal{N}_D} :\ g \in \mathsf{LOG}_D \rangle,$$

which is a subgroup of $\mathsf{SL}(\mathcal{N}_D)$.

We remark that Topaz is the name of some gemstone. This name may come from "tapas", a Sanskrit (the ancient language of India) word which means "fire". Tapas is now more well-known as a wide variety of snacks in Spanish cuisine, which is designed to encourage conversation.

It is noteworthy that the classification of the phase spaces of the lit-only σ-game on loopless graphs, namely Table 3.1, is well accompanied by a classification of the lit-only groups and Topaz groups. To give more details of this, we need some facts about classical groups and so we should turn to quadratic forms and bilinear forms now.

Let $\mathcal{V}$ be a vector space over the binary field $\mathbb{F}_2$ and let $b :\ \mathcal{V} \times \mathcal{V} \to \mathbb{F}_2$ be a bilinear map. We call b a *symmetric bilinear form* on $\mathcal{V}$ if $b(x, y) \equiv b(y, x)$ for all $x, y \in \mathcal{V}$ and we call b an *alternating bilinear form* on $\mathcal{V}$ if $b(x, x) \equiv 0$ for all $x \in \mathcal{V}$. The bilinear form b is *non-degenerate* if $x = 0$ is the only possible $x \in \mathcal{V}$ such that $b(x, y) \equiv 0$ for all $y \in \mathcal{V}$, or equivalently, the matrix for the bilinear map b with respect to any basis of $\mathcal{V}$ is nonsingular. Since we are working on a field of characteristic 2, an alternating form must be symmetric, but not vice versa. A *symplectic form* is a nondegenerate alternating bilinear form.

For any map $Q :\ \mathcal{V} \to \mathbb{F}_2$, we define its *polarisation* to be the map $\nabla_Q :\ \mathcal{V} \times \mathcal{V} \to \mathbb{F}_2$ such that

$$\nabla_Q(x, y) :\equiv Q(x + y) - Q(x) - Q(y), \quad \forall x, y \in \mathcal{V}.$$

A *quadratic form* on $\mathcal{V}$ is a map $Q :\ \mathcal{V} \to \mathbb{F}_2$ such that ∇_Q is bilinear, and so an alternating form. We call a quadratic form Q *non-degenerate* or *non-defective* if $x = 0$ is the only $x \in \mathcal{V}$ such that $Q(x) = 0$ and $\nabla_Q(x, y) = 0$ for all $y \in \mathcal{V}$; equivalently, there is no $\mathbb{F}_2$-linear map p from $\mathcal{V}$ to one of its proper subspaces such that $Q = Q \circ p$ [13, Theorem 7.3].

Two bilinear forms b and b' on the binary linear spaces $\mathcal{V}$ and $\mathcal{V}'$ respectively are *isometric* provided there is an $\mathbb{F}_2$-linear isomorphism $\rho\colon \mathcal{V} \to \mathcal{V}'$ such that $b = b' \circ (\rho, \rho)$. Two quadratic forms Q and Q' on the binary linear spaces $\mathcal{V}$ and $\mathcal{V}'$ respectively are *isometric* provided there is an $\mathbb{F}_2$-linear isomorphism $\rho\colon \mathcal{V} \to \mathcal{V}'$ such that $Q = Q' \circ \rho$.

To illuminate the definitions above, let us pause for some simple examples. Let X be a set. All subsets of X of even size, denoted by $\binom{X}{\text{even}}$, is a vector subspace of 2^X. It is clear that $|\cdot|/2$ is a quadratic form on the vector space $\binom{X}{\text{even}}$ and its polarisation is given by $\nabla_{|\cdot|/2}(x, y) \equiv |x \cap y|$ for all $x, y \in \binom{X}{\text{even}}$. Note that the quotient space $2^X/\{\mathbf{0}, X\}$ is also a binary linear space. We often write an element $\{A, B\} \in 2^X/\{\mathbf{0}, X\}$ as $A|B$ and thus view $2^X/\{\mathbf{0}, X\}$ as the set of splits on X, the notation $A|B$ meaning that X is partitioned into two disjoint sets A and B. For each $S \in 2^X/\{\mathbf{0}, X\}$, we define $\overline{S_x}$ and S_x to be the two sets such that $x \in S_x$, $x \notin \overline{S_x}$ and $S = S_x|\overline{S_x}$. For any $x \in X$, define the quadratic form Q_x on $2^X/\{\mathbf{0}, X\}$ by setting

$$Q_x(S) :\equiv |\overline{S_x}|.$$

Note that the polarisation of Q_x vanishes everywhere. We point out that there is a natural nondegenerate pairing of $\binom{X}{\text{even}}$ and $2^X/\{\mathbf{0}, X\}$ as vector spaces [15]:

$$\begin{array}{ccc} \binom{X}{\text{even}} \times 2^X/\{\mathbf{0}, \mathrm{V}_G\} & \to & \mathbb{F}_2, \\ (Z, A|B) & \mapsto & |A \cap Z|. \end{array}$$

Consider a binary vector space $\mathcal{V}$ and a subgroup H of $\mathsf{GL}(\mathcal{V})$. We say that H *preserves* a bilinear form b on $\mathcal{V}$ if $b \circ (g, g) = b$ for all $g \in H$ and we say that H *preserves* a quadratic form q on $\mathcal{V}$ if $q \circ g = q$ for all $g \in H$.

For a digraph D, its adjacency form $\mathbb{A}_D$ is a bilinear form on 2^{V_D} given by

$$\mathbb{A}_D(x, y) :\equiv \sum_{uv \in \mathrm{A}_D} u^*(x)v^*(y).$$

If D is loopless and symmetric, then $\mathbb{A}_D$ is an alternating form.

Lemma 4.6 ([52]). *Let D be a loopless strongly connected digraph. The following statements hold.*

(i) *If D is asymmetric, then TG_D does not preserve any symplectic form on $\mathcal{N}_D$.*

(ii) *If D is symmetric, then TG_D preserves a unique symplectic form ω_D on the vector space $\mathcal{N}_D$, where ω_D is defined uniquely by*

$$\omega_D \circ (\mathrm{N}_D, \mathrm{N}_D) = \mathbb{A}_D\,.$$

Take a graph G. Regarding G as a one-dimensional abstract simplicial complex, we can define its *Euler characteristic* (over $\mathbb{F}_2$), denoted by $\chi(G)$, as $|\,\mathrm{V}_G\,|-|\,\mathrm{E}_G\,| \in$

$\mathbb{F}_2$. The *Euler form* of G (over $\mathbb{F}_2$), denoted by χ_G, is the quadratic form on 2^{V_G} given by

$$\chi_G(x) :\equiv \chi(G[x]) \equiv \sum_{v\in \mathrm{V}_G \setminus \mathrm{L}_G} v^*(x)^2 - \sum_{uv\in \mathrm{E}_G \setminus \mathrm{L}_G} u^*(x)v^*(x) \pmod 2,$$

for all $x \in 2^{\mathrm{V}_G}$, where $G[x]$ is the subgraph of G induced by $x \subseteq \mathrm{V}_G$.

Definition 4.7. A graph G is *polished* provided there exists a (unique) quadratic form q_G on $\mathcal{N}_G$ such that

$$q_G \circ \mathrm{N}_G = \chi_G. \tag{4.3}$$

A graph is *unpolished* if it is not polished.

Lemma 4.8 ([52])**.** *Let G be a loopless connected graph. The following statements hold.*

(i) *If G is unpolished, then TG_G does not preserve any non-zero quadratic form on the vector space $\mathcal{N}_G$.*

(ii) *If G is polished, then TG_G preserves a unique non-zero quadratic form q_G on vector space $\mathcal{N}_G$, where q_G is defined uniquely by Eq. (4.3).*

We use algebraic properties to define polished graphs. Is there any geometric characterization of them? Here is a small step in this direction.

Theorem 4.9 ([52])**.** *Let G be a loopless connected multigraph. Then, $\mathfrak{L}(G)$ is polished if and only if $|\,\mathrm{V}_G\,| \not\equiv 2 \pmod 4$.*

To proceed, let us review some relevant basic facts about classical groups [28, 46]. Let $\mathcal{V}$ be an $\mathbb{F}_2$-linear space. The *orthogonal group* on $\mathcal{V}$ with respect to a quadratic form q on $\mathcal{V}$, denoted by $\mathsf{O}(\mathcal{V}, q)$, is the subgroup of $\mathsf{SL}(\mathcal{V})$ preserving q, namely,

$$\mathsf{O}(\mathcal{V}, q) := \{g \in \mathsf{SL}(\mathcal{V}) :\ q \circ g = q\}.$$

The *symplectic group* on $\mathcal{V}$ with respect to a symplectic form ω on $\mathcal{V}$ is the subgroup of $\mathsf{SL}(\mathcal{V})$ preserving ω, namely,

$$\mathsf{Sp}(\mathcal{V}, \omega) := \{g \in \mathsf{SL}(\mathcal{V}) :\ \omega \circ (g, g) = \omega\}.$$

It is clear that $\mathsf{O}(\mathcal{V}, q) \le \mathsf{Sp}(\mathcal{V}, \nabla_q)$.

Let us first consider the case that $\dim \mathcal{V} = 2k + 1$ is odd. In this case, there is no symplectic form on $\mathcal{V}$ and there is a unique nondegenerate quadratic form q_{2k+1} on $\mathcal{V}$ up to isometry. We write O_{2k+1} for $\mathsf{O}(\mathcal{V}, q_{2k+1})$, which is determined by k up to isomorphism and does not depend on the choice of q_{2k+1}.

We next turn to the case that $\dim \mathcal{V} = 2k$ is even. In this case, a quadratic form Q on $\mathcal{V}$ is non-degenerate if and only if the alternating form ∇_Q is non-degenerate, namely if and only if ∇_Q is symplectic. Up to isometry, there is a unique symplectic form on $\mathcal{V}$, hence a unique symplectic group on $\mathcal{V}$ up to isomorphism. Let

$$\mathsf{Sp}_{2k} := \mathsf{Sp}(\mathcal{V}, \omega),$$

where ω is a symplectic form on $\mathcal{V}$, and call it the symplectic group of dimension $2k$. We remark that $\mathsf{O}_{2k+1} = \mathsf{Sp}_{2k}$ [28, Theorem 14.2] and the order of the group is given in [28, Theorem 3.12]. Up to isometry, there are exactly two non-degenerate quadratic forms on $\mathcal{V}$. If Q is one of them, i.e., if ∇_Q is symplectic, the *Arf invariant* of it, named after the Turkish mathematician Cahit Arf and referred to be the "democratic invariant" by William Browder [10], is the element of $\mathbb{F}_2$ that occurs most often among the values of Q. We can index the two nondegenerate quadratic forms on $\mathcal{V}$ by q^+_{2k}, which has Arf invariant 0, and q^-_{2k}, which has Arf invariant 1 [28, Theorem 13.14]. They indeed give rise to two nonisomorphic orthogonal groups [28, Theorem 14.48],

$$\mathsf{O}^+_{2k} := \mathsf{O}(\mathcal{V}, q^+_{2k}) \quad \text{and} \quad \mathsf{O}^-_{2k} := \mathsf{O}(\mathcal{V}, q^-_{2k}).$$

A formula for the sizes of the two groups is reported as [28, Theorem 14.48]. We call O^+_{2k} and O^-_{2k} the dimension-$2k$ orthogonal groups of plus type and minus type respectively. Note that $i \mapsto (-1)^i$ gives an isomorphism from the additive group $\mathbb{F}_2$ to the multiplicative group $\{+1, -1\}$ and this explains the correspondence between the Arf invariants and the types of the orthogonal groups.

A line graph is *opal* if it has a root multigraph which has an odd number of vertices and a line graph is *emerald* if it has a root multigraph which has an even number of vertices. A line graph may be both opal and emerald. Actually, it can be shown that loopless connected opal emerald line graphs are exactly complete 3-partite graphs and complete bipartite graphs except P_2, which is opal but not emerald, and all loopless connected opal emerald line graphs have rank 2. Thus, graph classes in Theorem 4.10 are disjoint and form a partition of all strongly connected loopless nonempty digraphs.

Theorem 4.10 ([52])**.** *Let D be a strongly connected loopless nonempty digraph. The topaz group TG_D and lit-only group LOG_D for symmetric digraphs are determined by the type of D as follows.*

Graph class	TG_D	LOG_D
Opal line graph	$\mathsf{Sym}_{\mathrm{rank}_D+1}$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\mathrm{corank}_D}$
Emerald line graph of rank at least 4	$\mathsf{Sym}_{\mathrm{rank}_D+2}$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\mathrm{corank}_D-1}$
Polished cuspidal graph	$\mathsf{O}^+(\mathcal{N}_D)$ *or* $\mathsf{O}^-(\mathcal{N}_D)$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\mathrm{corank}_D}$
Unpolished cuspidal graph	$\mathsf{Sp}(\mathcal{N}_D)$	$\mathsf{TG}_D \ltimes \mathcal{N}_D^{\mathrm{corank}_D-1}$
Asymmetric digraph	$\mathsf{SL}(\mathcal{N}_D)$	

One day in the winter of 2012 and on the second floor of our old mathematics building at SJTU, Eiichi Bannai, one of our colleagues at SJTU then who had been sitting in several of our talks on the lit-only σ-game on different occasions, posed to us his key question: "Is there any group theory in your work"? We do not know much group theory and we even do not write down any definition of a group in § 2, although we do understand that group and symmetry are often viewed as the core of mathematics. Henceforth, what we could do is to explain to him in some way our partial work of determining the topaz group and lit-only group. Eiichi firmly claimed that it should be some easy job to get a complete solution of our "group theory" question and suggested us to loop up some old work by McLaughlin [39, 40]. This really calls up our courage to reach Theorem 4.10 in a short time. We conjectured that the lit-only group for a loopless strongly connected asymmetric digraph D is $\mathsf{TG}_D \ltimes \mathcal{N}_D^{\mathrm{corank}_D}$. Maybe, it can also be verified or disproved easily?

4.3. **Nuclei.** A *nucleus* of a digraph M is a nonsingular vertex-induced subgraph N of M fulfilling $\operatorname{rank} N = \operatorname{rank} M$. The set of all nuclei of M is denoted by $\mathfrak{N}(M)$. If M is a nonsingular digraph, $\mathfrak{N}(M)$ consists of M itself. For example, nuclei of a complete bipartite graph are its set of vertex-induced P_2's.

Theorem 4.11 ([52])**.** *Let G be a connected graph and H be a nonempty connected subgraph of G. The digraph $\mathfrak{L}(H)$ is a connected nucleus of $\mathfrak{L}(G)$ if and only if one of the following holds:*

- (i) $\mathrm{L}_G \neq \emptyset$, *and H is the union of a spanning tree of G and a loop of G;*
- (ii) $\mathrm{L}_G = \emptyset$ *and $|\,\mathrm{V}_G\,|$ is odd, and H is a spanning tree of G;*
- (iii) $\mathrm{L}_G = \emptyset$ *and $|\,\mathrm{V}_G\,|$ is even, and H is a spanning tree of $G - v$ for some $v \in \mathrm{V}_G$.*

A graph class is *connected-hereditary* if it is closed under taking connected vertex-induced subgraphs, is *hereditary* if it is closed under taking vertex-induced subgraphs, and is *additive* if it is closed under taking disjoint unions.

Theorem 4.12 ([52])**.** *Let $\mathscr{G}$ and $\mathscr{H}$ be two connected-hereditary graph classes. If every minimal graph in $\mathscr{G} \setminus \mathscr{H}$ is nonsingular, then every connected graph in $\mathscr{H} \setminus \mathscr{G}$ has a connected nucleus in $\mathscr{G} \setminus \mathscr{H}$.*

Corollary 4.13 ([52])**.** *Let $\mathscr{G}$ be an additive and hereditary graph class. Every (connected) graph in $\mathscr{G}$ has a (connected) nucleus in $\mathscr{G}$.*

Corollary 4.14 ([52])**.** *For each of the following graph classes, every graph in it has a connected nucleus in the same graph class.*

- $\{$*Connected loopless line graphs*$\} = \{$*Connected loopless line graphs*$\} \setminus \emptyset$.
- $\{$*Connected loop-linked line graphs*$\} = \{$*Connected line graphs*$\}\setminus\{$*Loopless graphs*$\}$.

- $\{$*Connected loopless cuspidal graphs*$\} = \{$*Connected loopless graphs*$\} \backslash \{$*Line graphs*$\}$.
- $\{$*Connected loop-linked cuspidal graphs*$\} = \{$*Connected graphs*$\} \backslash \{$*Loopless graphs or line graphs*$\}$.

Corollary 4.15 ([52])**.** *All loopless nonsingular graphs are polished. In particular, nuclei of loopless graphs are polished.* □

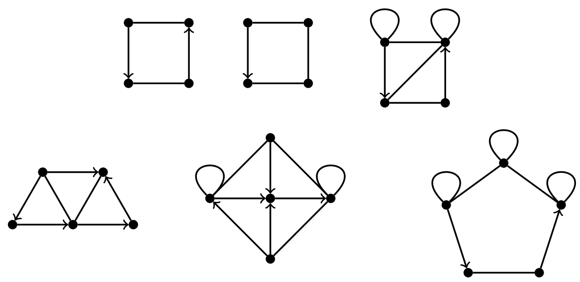


Figure 4.10. Some strongly connected digraphs which do not have strongly connected nuclei.

In Fig. 4.10, we display some strongly connected digraphs which do not have any strongly connected nuclei.

Problem 4.16. When does a strongly connected digraph have a strongly connected nucleus?

5. Origin of eggs

As a player of the lit-only σ-game, you may be wondering how exactly does our goose lay those strange eggs as displayed in § 4. To convince you that the eggs are not stolen from elsewhere, let us indicate below our approaches in getting Theorems 3.4 and 4.10.

Firstly, here is our strategy for loopless graphs.

Step 1 Understand the structures of line graphs and cuspidal graphs. § 4.1 presents some such work. Thoughout the whole process of our project, we need to apply various results established here and this step is really the foundation for everything.

Step 2 Analyze various forms attached to graphs, say symplectic form and Euler form as discussed in § 4.2. The analysis of these forms produces some rough descriptions of topaz group, lit-only group, and dynamical behavior of the lit-only σ-game.

Step 3 Classify the topaz group. The work in [40] suggests to us a list of candidates for a more careful inspection.

Step 4 Classify the phase spaces of nonsingular loopless graphs. The phase spaces for nonsingular graphs can be thought as the orbits of the topaz groups acting on the phase spaces regarded as a set. Thus, after determining all topaz groups, which are classical groups, we can get a description of the phase spaces with the help of known results about classical groups.

Step 5 Classify the phase spaces of all loopless graphs. In this step, we reduce the problem to the problem for nonsingular graphs, thanks to the concept of nuclei as introduced in § 4.3.

Step 6 Classify the lit-only group. In this step, we use the description of the phase space to prove that the lit-only group is a semidirect product of the topaz group and some abelian group.

Next, we need to deal with graphs with loops. Unfortunateneely, we do not have the blessings of groups any more. We somehow try to replace those group theoretic arguments for the previous case by some complicated combinatorial arguments and can thus study the structure of the phase space directly.

During August 12–25, 2019, the International Conference and PhD-Master Summer School on Groups and Graphs, Designs and Dynamics [5] was held in Yichang, China. The younger author of this essay gave a talk there entitled "Root systems over $\mathbb{F}_2$," which is basically to report the tour above towards Theorems 3.4 and 4.10. We assume that you have seen groups, graphs and dynamics from our story. Is design missing from the talk? No, it is not lost but already seen from the title of the talk. We mention that the connected components in the phase space of the lit-only σ-game can be thought of as root systems over $\mathbb{F}_2$. To explain briefly the connection between the usual root systems and our binary root system, let us end this section with the following dictionary.

Root system over $\mathbb{R}$	Root system over $\mathbb{F}_2$
Cartan matrix	Adjacency matrix
Simple roots	Vertices
Dynkin digram	Graph
Simple reflection	Simple transvection
Weyl group	Lit-only group
Euclidean space	Phase space
Root system	Connected component in the phase space

Table 5.1. A comparison between root systems over $\mathbb{R}$ and over $\mathbb{F}_2$.

[5] http://math.sjtu.edu.cn/conference/G2D2/

6. A bite to go

We will be described as a bit of an odd duck if we tell you so many stories about the lit-only σ-game but say nothing about the status of Problem 2.1. So, here is our brief answer to that problem: Yes, a polynomial time algorithm does exist for symmetric digraphs.

According to Theorem 3.4, given a connected symmetric digraph D and two subsets $x, y \subseteq \mathrm{V}_D$, we can determine whether or not $x \to_D y$ if we can

(i) find the graph class in Table 3.1 that contains D;
(ii) find the strongly connected components C_x and C_y in $\mathcal{PS}_D$ containing x and y, respectively.

The first step to determine the graph class of D is to recognize if D is a line graph or not. When D is reduced, we give a linear time algorithm to recognize line graphs and reconstruct their root graphs in [52]. When D is not reduced, we first find the modular decomposition of D, which can be constructed in linear time, and then we can read its reduced graph $\widetilde{D}$ from its modular decomposition. The graph D is a line graph if and only if its reduced graph $\widetilde{D}$ is a line graph, and the root multigraphs of D can be obtained from root graphs of $\widetilde{D}$ easily. Once we can determine whether or not D is a line graph, we can find the graph class in Table 3.1 that contains D by using corresponding linear algebraic characterization in [52], which can be done in polynomial time.

Finding the strongly connected components C_x and C_y requires explicit description of all strongly connected components in $\mathcal{PS}_D$. We refer you to [52] for more details.

To finish our short but full journey about the lit-only σ-game, let us try a small bite of some piece of the claimed polynomial time algorithm: the reconstruction of a root graph of the given graph G in Fig. 6.1. We first pick a maximal clique $\{x, y, z\}$ of G. For each vertex in the clique, say x, we choose the subgraph induced by x and its neighbourhood $\{y, z, b, c\}$ that are not in $\{x, y, z\}$. Then, we get another clique $\{x, b, c\}$. We continue this procedure for all encountered cliques and all of their vertices. At the end, we get four cliques: $\{x, y, z\}, \{x, b, c\}, \{z, a, b\}, \{y, a, c\}$. The intersection graph of these four cliques is the clique K_4, whose line graph turns out to be G. For general graphs, we can use the same idea to find good partitions of the edges sets, if they exist. A root graph can be constructed in linear time for any given such good partition.

As suggested by Friedrich Nietzsche, if you would go up high, then use your own legs! Hopefully, the bit of mathematics we have discussed so far will inspire some readers and even motivate some brave kids have an after-dinner walk further into the graden of discrete-time graph processes.

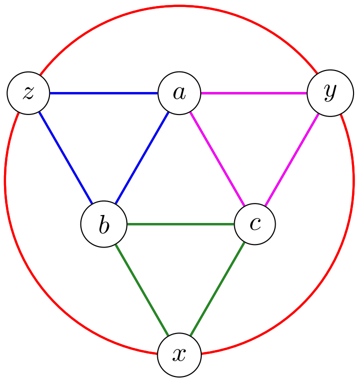


Figure 6.1. A line graph and its edge clique partition.

7. Acknowledgements

Yaokun Wu dedicates this essay to his father, Jinbao Wu (May 31st, 1936 – Jan. 24th, 2017), for his love and tolerance. When he was a kid, his father challenged him with puzzles on counterfeit coins and others; but he never succeeded in solving any such puzzles even at the end of his childhood. According to A.P.J. Abdul Kalam (again!), E.N.D means "Effort Never Dies". Seeing that many smart and kind persons have been around in different periods to help, the father of Yaokun Wu should rest assured that his son will enjoy playing many nice games as he wishes. This work was supported by NSFC (11671258) and STCSM (17690740800). Ziqing Xiang was partially supported by the National Science Foundation grant DMS-1344994.

References

[1] A. Abdollahi and M. Zallaghi. Character sums for Cayley graphs. *Communications in Algebra*, 43(12):5159–5167, 2015. ISSN 0092-7872. DOI: 10.1080/00927872.2014.967398.

[2] M. A. Armstrong. *Groups and Symmetry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1988. ISBN 0-387-96675-7. DOI: 10.1007/978-1-4757-4034-9.

[3] B. Bagchi and N. S. Narasimha Sastry. Even order inversive planes, generalized quadrangles and codes. *Geometriae Dedicata*, 22(2):137–147, 1987. ISSN 0046-5755. DOI: 10.1007/BF00181262.

[4] R. B. Bapat. Challenges and frustrations of being a mathematician. *The Mathematics Student*, 76(1–4):49–54, 2007. URL http://www.isid.ac.in/~rbb/imsgen.pdf.

[5] R. Barua and S. Ramakrishnan. σ-game, σ^+-game and two-dimensional additive cellular automata. *Theoretical Computer Science*, 154(2):349–366, 1996. ISSN 0304-3975. DOI: 10.1016/0304-3975(95)00091-7.

[6] P. Batra. Vogan diagrams of real forms of affine Kac–Moody Lie algebras. *Journal of Algebra*, 251(1):80–97, 2002. ISSN 0021-8693. DOI: 10.1006/jabr.2001.9118.

[7] L. W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9(2):129–135, 1970. DOI: 10.1016/S0021-9800(70)80019-9.

[8] A. Borel and J. de Siebenthal. Les sous-groupes fermés de rang maximum des groups de Lie clos. *Commentarii Mathematici Helvetici*, 23(1):200–221, 1949. ISSN 0010-2571. DOI: 10.1007/BF02565599.

[9] M. Borovoi and Z. Evenor. Real homogeneous spaces, Galois cohomology, and Reeder puzzles. *Journal of Algebra*, 467:307–365, 2016. DOI: 10.1016/j.jalgebra.2016.07.032.

[10] W. Browder. The Kervaire invariant of framed manifolds and its generalization. *Annals of Mathematics. Second Series*, 90:157–186, 1969. ISSN 0003-486X. DOI: 10.2307/1970686.

[11] M.-K. Chuah. Cartan automorphisms and Vogan superdiagrams. *Mathematische Zeitschrift*, 273(3):793–800, 2012. ISSN 1432-1823. DOI: 10.1007/s00209-012-1030-z.

[12] M.-K. Chuah and C.-C. Hu. Extended Vogan diagrams. *Journal of Algebra*, 301(1):112–147, 2006. ISSN 0021-8693. DOI: 10.1016/j.jalgebra.2005.12.022.

[13] K. Conrad. Bilinear forms. *notes*, page 61 pp., 2015. URL http://www.math.uconn.edu/~kconrad/blurbs/linmultialg/bilinearform.pdf.

[14] R. T. Curtis and T. R. Morris. Codes. In L. Beineke and R. J. Wilson, editors, *Graph Connections*, chapter 8, pages 116–127. Oxford University Press, 1997.

[15] A. Dress and M. Steel. Mapping edge sets to splits in trees: the path index and parsimony. *Annals of Combinatorics*, 10(1):77–96, 2006. DOI: 10.1007/s00026-006-0275-9.

[16] H. Eriksson, K. Eriksson, and J. Sjöstrand. Note on the lamp lighting problem. *Advances in Applied Mathematics*, 27(2–3):357–366, 2001. DOI: 10.1006/aama.2001.0739.

[17] I. Z. Evenor. Solutions of Reeder's puzzle. *arXiv:1511.03069*, 2015. URL http://arxiv.org/abs/1511.03069.

[18] R. Fioresi and R. Palmieri. The push the button algorithm for contragredient Lie superalgebras. *arXiv:1602.06876*, 2016. URL http://arxiv.org/abs/1602.06876.

[19] M. Florence and F. Meunier. Completely symmetric configurations for σ-games on grid graphs. *Journal of Algebraic Combinatorics*, 31(4):533–545, 2010. ISSN 0925-9899. DOI: 10.1007/s10801-009-0199-7.

[20] A. S. Fraenkel. Virus versus mankind. In T. Marsland and I. Frank, editors, *Computers and Games*, pages 204–213. Lecture Notes in Computer Science, Springer, 2001. DOI: 10.1007/3-540-45579-5_13.

[21] A. S. Fraenkel. Mathematical chats between two physicists. In D. Wolfe and T. Rodgers, editors, *Puzzlers' Tribute: A Feast for the Mind, honoring Martin Gardner*, pages 315–325. A K Peters, 2002. URL http://www.wisdom.weizmann.ac.il/~fraenkel/Papers/wolf4.ps.

[22] A. S. Fraenkel and R. J. Nowakowski. Combinatorial games: selected short bibliography with a succinct gourmet introduction. In R. J. Nowakowski, editor, *Games of No Chance 4*, volume 63 of *Mathematical Sciences Research Institute Publications*, pages 309–339. Cambridge University Press, Cambridge, 2015. URL http://library.msri.org/books/Book63/files/131104-Fraenkel.pdf.

[23] M. Gintz. Classifying graph Lie algebras. Technical report, MIT, 2017. URL http://math.mit.edu/research/highschool/primes/materials/2017/Gintz.pdf.

[24] J. Goldwasser and W. F. Klostermeyer. Maximum orbit weights in the σ-game and lit-only σ-game on grids and graphs. *Graphs and Combinatorics*, 25(3):309–326, 2009. DOI: 10.1007/s00373-009-0841-0.

[25] J. Goldwasser, W. Klostermeyer, and G. Trapp. Characterizing switch-setting problems. *Linear and Multilinear Algebra*, 43(1-3):121–135, 1997. DOI: 10.1080/03081089708818520.

[26] J. Goldwasser, X. Wang, and Y. Wu. Minimum light numbers in the σ-game and lit-only σ-game on unicyclic and grid graphs. *The Electronic Journal of Combinatorics*, 18(1):P214, 2011. URL http://www.combinatorics.org/ojs/index.php/eljc/article/view/v18i1p214.

[27] J. L. Goldwasser, X. Wang, and Y. Wu. Does the lit-only restriction make any difference for the σ-game and σ^+-game? *European Journal of Combinatorics*, 30(4):774–787, 2009. ISSN 0195-6698. DOI: 10.1016/j.ejc.2008.09.020.

[28] L. C. Grove. *Classical Groups and Geometric Algebra*, volume 39 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002. ISBN 0-8218-2019-2.

[29] D. Grynkiewicz, V. F. Lev, and O. Serra. Connectivity of addition Cayley graphs. *Journal of Combinatorial Theory, Series B*, 99(1):202–217, 2009. ISSN 0095-8956. DOI: 10.1016/j.jctb.2008.06.003.

[30] V. Guruswami, D. Micciancio, and O. Regev. The complexity of the covering radius problem. *Computational Complexity*, 14(2):90–121, 2005. ISSN 1420-8954. DOI: 10.1007/s00037-005-0193-y.

[31] C.-I. Hsin. Graph painting and the Borel-de Siebenthal property. *Revista de la Unión Matemática Argentina*, 54(2):67–73, 2013. URL http://inmabb.criba.edu.ar/revuma/pdf/v54n2/v54n2a06.pdf.

[32] C.-I. Hsin. The generalized Borel-de Siebenthal theorem. *Quaestiones Mathematicae*, 38(4):529–534, 2015. DOI: 10.2989/16073606.2014.981732.

[33] H.-W. Huang. Two-lit trees for lit-only σ-game. *Linear Algebra and its Applications*, 438(3):1057–1066, 2013. DOI: 10.1016/j.laa.2012.08.033.

[34] J. Hughes. Random lights out processes on graphs. *Advances in Applied Mathematics*, 51(2):254–265, 2013. ISSN 0196-8858. DOI: 10.1016/j.aam.2013.03.002.

[35] G. Istrate. Reachability and recurrence in a modular generalization of annihilating random walks (and lights-out games) on hypergraphs. *Theoretical Computer Science*, 580:83–93, 2015. DOI: 10.1016/j.tcs.2015.02.035.

[36] M. Kassabov. Symmetric groups and expander graphs. *Inventiones Mathematicae*, 170(2):327–354, 2007. ISSN 0020-9910. DOI: 10.1007/s00222-007-0065-y.

[37] A. W. Knapp. A quick proof of the classification of simple real Lie algebras. *Proceedings of the American Mathematical Society*, 124(10):3257–3259, 1996. ISSN 0002-9939. DOI: 10.1090/S0002-9939-96-03448-X.

[38] C. H. Li. On isomorphisms of finite Cayley graphs—a survey. *Discrete Mathematics*, 256(1-2):301–334, 2002. ISSN 0012-365X. DOI: 10.1016/S0012-365X(01)00438-1.

[39] J. McLaughlin. Some groups generated by transvections. *Archiv der Mathematik*, 18(4):364–368, 1967. DOI: 10.1007/BF01898827.

[40] J. McLaughlin. Some subgroups of $\mathrm{SL}_n(\mathrm{F}_2)$. *Illinois Journal of Mathematics*, 13(1):108–115, 1969. DOI: 10.1215/ijm/1256053741.

[41] B. Ransingh. Vogan diagrams of affine twisted Lie superalgebras. *International Journal of Pure and Applied Mathematics*, 84(5):539–547, 2013. DOI: 10.12732/ijpam.v84i5.7.

[42] M. Ronan. *Symmetry and the Monster: One of the Greatest Quests of Mathematics*. Oxford University Press, Oxford, 2006. ISBN 978-0-19-280722-9; 0-19-280722-6.

[43] H. Rubenthaler. The Borel-de Siebenthal theorem, the classification of equi-rank groups, and related compact and semi-compact dual pairs. *Hokkaido Mathematical Journal*, 33(1):185–205, 2004. ISSN 0385-4035. DOI: 10.14492/hokmj/1285765999.

[44] J. Schicho and J. Top. Algebraic approaches to FlipIt. In D. Ellwood, H. Hauser, S. Mori, and J. Schicho, editors, *The Resolution of Singular Algebraic Varieties*, volume 20 of *Clay Mathematics Proceedings*, pages 319–325. American Mathematical Society, 2014. URL http://www.claymath.org/library/proceedings/cmip20.pdf.

[45] K. Sutner. Linear cellular automata and the Garden-of-Eden. *The Mathematical Intelligencer*, 11(2):49–53, 1989. DOI: 10.1007/BF03023823.

[46] Z. X. Wan. *Geometry of Classical Groups over Finite Fields*. Science Press, second edition, 2002.

[47] X. Wang and Y. Wu. The σ-game on trees: Covering radius and tree order. 2006. preprint.

[48] X. Wang and Y. Wu. Minimum light number of lit-only σ-game on a tree. *Theoretical Computer Science*, 381(1–3):292–300, 2007. DOI: 10.1016/j.tcs.2007.05.033.

[49] X. Wang and Y. Wu. Lit-only σ-game on pseudo-trees. *Discrete Applied Mathematics*, 158(17):1945–1952, 2010. DOI: 10.1016/j.dam.2010.08.009.

[50] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168, 1932. ISSN 0002-9327. DOI: 10.2307/2371086.

[51] Y. Wu. Lit-only sigma game on a line graph. *European Journal of Combinatorics*, 30(1):84–95, 2009. DOI: 10.1016/j.ejc.2008.02.003.

[52] Y. Wu and Z. Xiang. The lit-only σ-game. in preparation.

[53] Y. Wu, Z. Xu, and Y. Zhu. Strongly connected multivariate digraphs. *Electron. J. Combin.*, 24(1):Paper 1.47, 44, 2017. ISSN 1077-8926. URL https://www.combinatorics.org/ojs/index.php/eljc/article/view/v24i1p47.

[54] M. Zaidenberg. Periodic harmonic functions on lattices and points count in positive characteristic. *Central European Journal of Mathematics*, 7(3): 365–381, 2009. DOI: 10.2478/s11533-009-0029-0.

[55] X. Zhang and C. Wang. Solutions to all-colors problem on graph cellular automata. *Complexity*, 2019. DOI: 10.1155/2019/3164692. Article ID 3164692, 11 pages.

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