Mathematica code to solve a certain degree 10 Diophantine equation in 3 variables under some conditions - Ziqing Xiang

This is the Mathematica code for the joint work with Eiichi Bannai, Etsuko Bannai, Wei-Hsuan Yu and Yan Zhu entitled "Classification of spherical 2-distance {4, 2, 1}-design".

Section 3

Definition of Gegenbauer polynomial $Q_{n,4}(\xi)$.

```
In[1]:= Q[n_, 4, ξ_] := n (n + 6) / 24 ((n^2 + 6 n + 8) ξ^4 - 6 (n + 2) ξ^2 + 3);
```

Definition of F. Eq. (3.4).

```
In[2]:= F = Factor[Q[n, 4, 1] + k Q[n, 4, y / k] + (v - k - 1) Q[n, 4, (-y - 1) / (v - k - 1)] /.
          v → 1 / μ (k - x) (k - y) /. k → μ - x y /. n → (μ - x y) (μ - x y - x) (x + 1) / μ / (x - y)]
```

Out[2]=

$$\Big((x + x y - \mu) (y + x y - \mu)^2 \left(x^2 y + x^3 y + x^2 y^2 + x^3 y^2 + 5 x \mu - x^2 \mu - 6 y \mu - 2 x y \mu - 2 x^2 y \mu + \mu^2 + x \mu^2\right) \left(x^2 y + x^3 y + x^2 y^2 + x^3 y^2 + 3 x \mu - x^2 \mu - 4 y \mu - 2 x y \mu - 2 x^2 y \mu + \mu^2 + x \mu^2\right) \left(x^5 y^2 + 2 x^6 y^2 + x^7 y^2 - x^2 y^3 - 2 x^3 y^3 - x^4 y^3 + x^5 y^3 + 2 x^6 y^3 + x^7 y^3 - x^2 y^4 - 2 x^3 y^4 - x^4 y^4 - 5 x^4 y \mu - 7 x^5 y \mu - 2 x^6 y \mu - x y^2 \mu - 2 x^2 y^2 \mu + 3 x^3 y^2 \mu - x^4 y^2 \mu - 8 x^5 y^2 \mu - 3 x^6 y^2 \mu + 2 y^3 \mu + 4 x y^3 \mu - x^2 y^3 \mu - 3 x^3 y^3 \mu + 6 x^3 \mu^2 + 5 x^4 \mu^2 + x^5 \mu^2 - 2 x y \mu^2 - 7 x^2 y \mu^2 + 6 x^3 y \mu^2 + 10 x^4 y \mu^2 + 3 x^5 y \mu^2 + 3 y^2 \mu^2 + 8 x y^2 \mu^2 + 3 x^2 y^2 \mu^2 - 3 x \mu^3 - 6 x^2 \mu^3 - 4 x^3 \mu^3 - x^4 \mu^3 + y \mu^3 + x y \mu^3\right)\Big) \Big/ \left(24 (1 + x)^2 (x - y)^4 (x y - \mu)^2 \mu^4\right)$$

The four factors F_0, F_1, F_2, F_3 of F. Eqs. (3.5) - (3.8).

In[3]:=
```
F0 = (μ - x - x y) (μ - y - x y)^2 / (24 μ^4 (1 + x)^2 (x - y)^4 (μ - x y)^2);
F0
F1 = μ^2 + 5 μ x + μ^2 x - μ x^2 - 6 μ y - 2 μ x y + x^2 y - 2 μ x^2 y + x^3 y + x^2 y^2 + x^3 y^2 ;
Collect[F1, {μ, y}, Factor]
F2 = μ^2 + 3 μ x + μ^2 x - μ x^2 - 4 μ y - 2 μ x y + x^2 y - 2 μ x^2 y + x^3 y + x^2 y^2 + x^3 y^2;
Collect[F2, {μ, y}, Factor]
F3 = 3 μ^3 x + 6 μ^3 x^2 - 6 μ^2 x^3 + 4 μ^3 x^3 - 5 μ^2 x^4 + μ^3 x^4 - μ^2 x^5 - μ^3 y +
    2 μ^2 x y - μ^3 x y + 7 μ^2 x^2 y - 6 μ^2 x^3 y + 5 μ x^4 y - 10 μ^2 x^4 y + 7 μ x^5 y - 3 μ^2 x^5 y +
    2 μ x^6 y - 3 μ^2 y^2 + μ x y^2 - 8 μ^2 x y^2 + 2 μ x^2 y^2 - 3 μ^2 x^2 y^2 - 3 μ x^3 y^2 + μ x^4 y^2 -
    x^5 y^2 + 8 μ x^5 y^2 - 2 x^6 y^2 + 3 μ x^6 y^2 - x^7 y^2 - 2 μ y^3 - 4 μ x y^3 + x^2 y^3 + μ x^2 y^3 +
    2 x^3 y^3 + 3 μ x^3 y^3 + x^4 y^3 - x^5 y^3 - 2 x^6 y^3 - x^7 y^3 + x^2 y^4 + 2 x^3 y^4 + x^4 y^4;
Collect[F3, {μ, y}, Factor]
```

Out[4]= $\left((-x - x y + \mu)\,(-y - x y + \mu)^2\right) / \left(24\,(1+x)^2\,(x-y)^4\,\mu^4\,(-x y + \mu)^2\right)$

Out[6]= $x^2\,(1+x)\,y + x^2\,(1+x)\,y^2 + \left(-(-5+x)\,x - 2\,\left(3 + x + x^2\right)\,y\right)\,\mu + (1+x)\,\mu^2$

Out[8]= $x^2\,(1+x)\,y + x^2\,(1+x)\,y^2 + \left(-(-3+x)\,x - 2\,\left(2 + x + x^2\right)\,y\right)\,\mu + (1+x)\,\mu^2$

Out[10]= $-x^5\,(1+x)^2\,y^2 - (-1+x)\,x^2\,(1+x)^2\,\left(1+x+x^2\right)\,y^3 + x^2\,(1+x)^2\,y^4 + \left(x^4\,(1+x)\,(5+2x)\,y + x\,(1+x)\,\left(1+x-4x^2+5x^3+3x^4\right)\,y^2 + (1+x)\,\left(-2-2x+3x^2\right)\,y^3\right)\,\mu + \left(-x^3\,(2+x)\,(3+x) - x\,\left(-2-7x+6x^2+10x^3+3x^4\right)\,y + \left(-3-8x-3x^2\right)\,y^2\right)\,\mu^2 + \left(x\,(1+x)\,\left(3+3x+x^2\right) + (-1-x)\,y\right)\,\mu^3$

Verify the factorization.

In[11]:=
```
Simplify[F0 F1 F2 F3 - F]
```

Out[11]= 0

Section 5.2

Alternative definition of *n* and *v*. Eqs. (5.1) and (5.2).

In[12]:=
```
n = (x + 1) (μ - x y) (μ - x y - x) / (μ (-y + x));
v = n + 1 - (y n + μ - x y) / x;
```

Step 1

Assumption on *y* in Step 1. Eq. (5.3).

In[14]:=
```
yassum = y ≤ - (2 x^3 + 3 x^2 + 3 x + 2) || - (2 x^3 + 3 x^2 - 3 x - 3) ≤ y ≤ -1;
```

The computer proof for Step 1.

In[15]:=
```
Simplify[v > n (n + 3) / 2 || F3 > 0, x ≥ 1 && μ ≥ 1 && yassum]
```

Out[15]= True

Step 2

Definition of *a*. Eq. (5.4).

In[16]:=
```
usub = μ → - (x + a) y;
```

Definition of G_1.

In[17]:=
```
G1 = Factor[(F3 /. usub) / y^2];
Collect[G1, {y, x}, Factor]
```

Out[18]=
$$-6 a^2 x^3 - a (17 + 5 a) x^4 + \left(-12 - 17 a - a^2\right) x^5 - 2 (7 + 2 a) x^6 - 4 x^7 + \left(-a \left(1 - 2 a + 3 a^2\right) x - 2 a \left(-1 + a + 3 a^2\right) x^2 - 2 \left(-1 - 4 a + 12 a^2 + 2 a^3\right) x^3 + \left(8 - 31 a - 22 a^2 - a^3\right) x^4 - 2 \left(7 + 20 a + 3 a^2\right) x^5 - 12 (2 + a) x^6 - 8 x^7\right) y + \left((-2 + a) (-1 + a) a + \left(2 - 2 a - 5 a^2 + a^3\right) x - 2 (-1 + 7 a) x^2 - 6 (1 + a) x^3 - 4 x^4\right) y^2$$

The computer proof for Step 2(a).

In[19]:=
```
Simplify[(G1 /. y → 0) < 0, 2 ≤ x && -x < a]
```

Out[19]= True

The computer proof for Step 2(b).

In[20]:=
```
Simplify[(G1 /. y → -1) > 0, 2 ≤ x && -x < a]
```

Out[20]= True

The computer proof for Step 2(c).

In[21]:=
```
Simplify[(G1 /. y → - (2 x^3 + 3 x^2 + 3 x + 2)) > 0, 2 ≤ x && (-x < a ≤ -1 || 3 ≤ a)]
```

Out[21]= True

Step 3

Definition of *b*. Eq. (5.11).

In[22]:=
$$\text{ysub} = y \to -\left(2 x^3 + 3 x^2 + \frac{3}{2} (-1 + a) a x - \frac{3}{2} (-1 + a)^2 a + \frac{1}{4 x} 3 (-1 + a) a \left(2 - 4 a + 3 a^2\right) - \frac{1}{4 x^2} 3 (-1 + a) a^2 \left(3 - 6 a + 4 a^2\right) + \frac{1}{8 x^3} 3 (-1 + a) a \left(5 - 9 a + 16 a^2 - 20 a^3 + 11 a^4\right) + \frac{b}{x^4}\right);$$

Definition of G_2.

In[23]:=
```
G2 = Factor[G1 /. ysub];
Collect[G2, x, Factor]
```

Out[24]=

$$\frac{1}{8}\left(-150 a+336 a^2-330 a^3+738 a^4-1854 a^5+3291 a^6-4305 a^7+3603 a^8-1563 a^9+234 a^{10}+96 b-248 a b-176 a^2 b-8 a^3 b\right)+\frac{(-2+a)(-1+a) a b^2}{x^8}+\frac{1}{4 x^7} b\left(-30 a^2+129 a^3-291 a^4+483 a^5-585 a^6+453 a^7-192 a^8+33 a^9+8 b-8 a b-20 a^2 b+4 a^3 b\right)+\frac{1}{64 x^6}\left(450 a^3-3195 a^4+12033 a^5-32382 a^6+67608 a^7-112005 a^8+149265 a^9-160236 a^{10}+135054 a^{11}-84645 a^{12}+36369 a^{13}-9405 a^{14}+1089 a^{15}-480 a b+1824 a^2 b-1968 a^3 b-336 a^4 b+5040 a^5 b-10320 a^6 b+10224 a^7 b-4512 a^8 b+528 a^9 b+128 b^2-896 a b^2\right)+\frac{1}{64 x^5} 3\left(150 a^2-990 a^3+2781 a^4-4233 a^5+1572 a^6+10824 a^7-34845 a^8+62355 a^9-78174 a^{10}+71370 a^{11}-45939 a^{12}+19155 a^{13}-4389 a^{14}+363 a^{15}-160 a b+1632 a^2 b-4128 a^3 b+6400 a^4 b-7360 a^5 b+5152 a^6 b-1408 a^7 b-128 a^8 b-128 b^2-128 a b^2\right)+\frac{1}{32 x^4}\left(225 a^2-3015 a^3+13878 a^4-38736 a^5+78291 a^6-119925 a^7+140382 a^8-125874 a^9+84177 a^{10}-37521 a^{11}+7479 a^{12}+1431 a^{13}-792 a^{14}+528 a b-432 a^2 b-1392 a^3 b+3984 a^4 b-5664 a^5 b+4416 a^6 b-1440 a^7 b-128 b^2\right)-\frac{1}{32 x^3} a\left(315 a-477 a^2-3546 a^3+19206 a^4-51723 a^5+95265 a^6-129942 a^7+133326 a^8-100431 a^9+52119 a^{10}-16497 a^{11}+2385 a^{12}-128 b-32 a b+96 a^2 b\right)+\frac{1}{16 x^2} a\left(105 a-1098 a^2+4506 a^3-12120 a^4+24231 a^5-36246 a^6+40053 a^7-31977 a^8+17478 a^9-5805 a^{10}+873 a^{11}+256 b-64 a b-96 a^2 b\right)+\frac{1}{16 x}\left(-408 a^2+708 a^3+1134 a^4-6750 a^5+15636 a^6-22548 a^7+20841 a^8-12051 a^9+4077 a^{10}-639 a^{11}+160 b+192 a b-384 a^2 b-64 a^3 b\right)+\frac{1}{8}\left(-180 a+1089 a^2-2388 a^3+3714 a^4-4506 a^5+3417 a^6-1404 a^7+462 a^8-204 a^9-112 b-320 a b-48 a^2 b\right) x-\frac{3}{4}\left(-15 a-82 a^2+343 a^3-648 a^4+814 a^5-598 a^6+178 a^7+8 a^8+32 b+16 a b\right) x^2+\frac{1}{2}\left(30 a-27 a^2-84 a^3+249 a^4-354 a^5+276 a^6-90 a^7-16 b\right) x^3$$

The computer proof for Step 3(a).

In[25]:=
```
Simplify[(G2 /. b → -3994) > 0, 90 ≤ x && -1 ≤ a ≤ 3]
```

Out[25]= True

The computer proof for Step 3(b).

In[26]:=
```
Simplify[(G2 /. b → 64) < 0, 90 ≤ x && -1 ≤ a ≤ 3]
```

Out[26]= True

Step 4

Definition of m^2. Eq. (5.12).

```
In[27]:= mmsub = mm → n - (4 x^2 + 4 x - 2);
```

Definition of G_3.

```
In[28]:= G3 = Factor[mm /. mmsub /. usub /. ysub];
G3
```

Out[29]=

$$\left(16 a b + 8 a^2 b - 30 a^2 x + 69 a^3 x - 108 a^4 x + 141 a^5 x - 78 a^6 x - 27 a^7 x + 33 a^8 x + 16 b x + 8 a^2 b x - 30 a x^2 + 84 a^2 x^2 - 129 a^3 x^2 + 168 a^4 x^2 - 195 a^5 x^2 + 186 a^6 x^2 - 117 a^7 x^2 + 33 a^8 x^2 + 12 a^2 x^3 - 48 a^3 x^3 + 90 a^4 x^3 - 108 a^5 x^3 + 78 a^6 x^3 - 24 a^7 x^3 - 24 a x^4 + 48 a^2 x^4 - 60 a^3 x^4 + 72 a^4 x^4 - 54 a^5 x^4 + 18 a^6 x^4 - 16 a x^5 + 24 a^2 x^5 - 24 a^3 x^5 + 36 a^4 x^5 - 12 a^5 x^5 - 16 a x^6 + 48 a^2 x^6 - 12 a^3 x^6 + 12 a^4 x^6 + 40 a^2 x^7 + 16 a^2 x^8\right) \Big/ \left((a + x) \left(8 b - 15 a x + 42 a^2 x - 75 a^3 x + 108 a^4 x - 93 a^5 x + 33 a^6 x + 18 a^2 x^2 - 54 a^3 x^2 + 60 a^4 x^2 - 24 a^5 x^2 - 12 a x^3 + 36 a^2 x^3 - 42 a^3 x^3 + 18 a^4 x^3 - 12 a x^4 + 24 a^2 x^4 - 12 a^3 x^4 + 8 x^5 - 12 a x^5 + 12 a^2 x^5 + 24 x^6 + 16 x^7\right)\right)$$

Definition of $\tilde{m}^2$. Eq. (5.12).

In[30]:=

$$\text{mmtsub} = \text{mmt} \to a^2 - \frac{(-1 + a)\, a^2}{x} + \frac{(-1 + a)\, a\, (1 + a^2)}{x^2} - \frac{1}{2 x^3} (-1 + a)\, a\, (1 + 2 a + 2 a^3) + \frac{1}{4 x^4} (-1 + a)\, a\, (7 - a + 4 a^2 + 4 a^4) + \frac{c}{x^5};$$

Definition of G_4.

```
In[31]:= G4 = Factor[mmt /. mmtsub];
Collect[G4, x, Factor]
```

Out[32]=

$$a^2 + \frac{c}{x^5} + \frac{1}{4 x^4} (-1 + a)\, a\, (7 - a + 4 a^2 + 4 a^4) - \frac{1}{2 x^3} (-1 + a)\, a\, (1 + 2 a + 2 a^3) + \frac{(-1 + a)\, a\, (1 + a^2)}{x^2} - \frac{(-1 + a)\, a^2}{x}$$

The computer proof for Step 4(a).

```
In[33]:= Simplify[G3 > (G4 /. c → -1620), 90 ≤ x && -1 ≤ a ≤ 3 && -3994 ≤ b ≤ 64]
Out[33]= True
```

The computer proof for Step 4(b).

```
In[34]:= Simplify[G3 < (G4 /. c → 3), 90 ≤ x && -1 ≤ a ≤ 3 && -3994 ≤ b ≤ 64]
Out[34]= True
```

The computer proof for Step 4(c).

```
In[35]:= Simplify[G4 < 9, 90 ≤ x && -1 ≤ a ≤ 3 && -1620 ≤ c ≤ 3]
Out[35]= True
```

Step 5

Definition of $\tilde{n}$. Eq. (5.16).

In[36]:=
```
nt = mmt + (4 x^2 + 4 x - 2);
Collect[nt /. usub /. ysub /. mmtsub, x, Factor]
```

Out[37]=
$$-2+a^2+\frac{c}{x^5}+\frac{1}{4x^4}(-1+a)\,a\left(7-a+4a^2+4a^4\right)-\frac{1}{2x^3}(-1+a)\,a\left(1+2a+2a^3\right)+\frac{(-1+a)\,a\left(1+a^2\right)}{x^2}-\frac{(-1+a)\,a^2}{x}+4x+4x^2$$

Definition of $\tilde{v}$. Eq. (5.16).

In[38]:=
```
vt = nt + 1 - (y nt + μ - x y) / x;
Collect[vt /. usub /. ysub /. mmtsub, x, Factor]
```

Out[39]=
$$\frac{1}{2}(-1+a)\left(2+18a-39a^2+25a^3\right)+\frac{bc}{x^{10}}+\frac{1}{8x^9}(-1+a)\,a\left(14b-2ab+8a^2b+8a^4b+15c-27ac+48a^2c-60a^3c+33a^4c\right)+\frac{1}{32x^8}(-1+a)\,a\left(-105a+309a^2-627a^3+999a^4-1119a^5+924a^6-705a^7+564a^8-372a^9+132a^{10}-16b-32ab-32a^3b-72ac+144a^2c-96a^3c\right)-\frac{1}{16x^7}(-1+a)\,a\left(-15a-51a^2+207a^3-345a^4+429a^5-471a^6+438a^7-306a^8+114a^9-16b-16a^2b-24c+48ac-36a^2c\right)+\frac{1}{16x^6}(-1+a)\,a\left(-72a+198a^2-351a^3+504a^4-585a^5+546a^6-390a^7+150a^8-16ab+24c-24ac\right)+\frac{1}{8x^5}\left(15a^2-87a^3+222a^4-375a^5+489a^6-483a^7+306a^8-87a^9-16b-8ab+8a^2b+8c-12ac+12a^2c\right)+\frac{1}{8x^4}\left(16a-26a^2-4a^3+74a^4-254a^5+425a^6-330a^7+99a^8+16b+24c\right)+\frac{1}{4x^3}\left(-34a+53a^2-73a^3+157a^4-217a^5+147a^6-33a^7+16b+8c\right)+\frac{1}{4x^2}(-1+a)\,a\left(30-74a+156a^2-195a^3+95a^4\right)-\frac{1}{2x}(-1+a)\,a\left(-4+33a-60a^2+34a^3\right)-2\left(1+6a-10a^2+4a^3\right)x+2\left(3-4a+4a^2\right)x^2+16x^3+8x^4$$

Definition of $\tilde{z}$. Eq. (5.15).

In[40]:=
```
zt = 144 mmt - (3 vt + (8 y + 4 x^3 + 6 x^2 + 3) (2 x + 1) - 3 / 2 mmt (mmt - 7)) ^2;
```

Definition of G_5.

In[41]:=
```
G5 = Factor[zt /. usub /. ysub /. mmtsub];
Collect[G5, x, Factor]
```

Out[42]=

$$-\frac{9\ (2\ b-c)^2\ c^2}{4\ x^{20}}-\frac{1}{8\ x^{19}}$$

$$9\ (-1+a)\ a\ (2\ b-c)\ c\ \left(14\ b-2\ a\ b+8\ a^2\ b+8\ a^4\ b+c-25\ a\ c+40\ a^2\ c-60\ a^3\ c+25\ a^4\ c\right)-$$

$$\frac{1}{64\ x^{18}}9\ (-1+a)\ a\ \left(-196\ a\ b^2+252\ a^2\ b^2-284\ a^3\ b^2+260\ a^4\ b^2-320\ a^5\ b^2+320\ a^6\ b^2-\right.$$

$$160\ a^7\ b^2+128\ a^8\ b^2-64\ a^9\ b^2+64\ a^{10}\ b^2-252\ a\ b\ c+1716\ a^2\ b\ c-4164\ a^3\ b\ c+$$

$$7212\ a^4\ b\ c-7992\ a^5\ b\ c+6432\ a^6\ b\ c-5160\ a^7\ b\ c+4128\ a^8\ b\ c-2784\ a^9\ b\ c+$$

$$864\ a^{10}\ b\ c-64\ b^2\ c-128\ a\ b^2\ c-128\ a^3\ b^2\ c+111\ a\ c^2-441\ a^2\ c^2+357\ a^3\ c^2+957\ a^4\ c^2-$$

$$4692\ a^5\ c^2+9012\ a^6\ c^2-10\,320\ a^7\ c^2+7536\ a^8\ c^2-2913\ a^9\ c^2+393\ a^{10}\ c^2+96\ b\ c^2-$$

$$\left.96\ a\ b\ c^2+576\ a^2\ b\ c^2-192\ a^3\ b\ c^2-32\ c^3+80\ a\ c^3-288\ a^2\ c^3+128\ a^3\ c^3\right)-\frac{1}{128\ x^{17}}$$

$$9\ (-1+a)\ a\ \left(784\ a^2\ b-4340\ a^3\ b+12\,280\ a^4\ b-24\,880\ a^5\ b+37\,466\ a^6\ b-44\,644\ a^7\ b+\right.$$

$$47\,092\ a^8\ b-46\,760\ a^9\ b+42\,682\ a^{10}\ b-33\,024\ a^{11}\ b+21\,744\ a^{12}\ b-13\,584\ a^{13}\ b+8032\ a^{14}\ b-$$

$$3776\ a^{15}\ b+928\ a^{16}\ b+224\ a\ b^2+192\ a^2\ b^2-352\ a^3\ b^2+640\ a^4\ b^2-640\ a^5\ b^2+448\ a^6\ b^2-$$

$$512\ a^7\ b^2+256\ a^8\ b^2-256\ a^9\ b^2+56\ a^2\ c-1702\ a^3\ c+10\,592\ a^4\ c-36\,980\ a^5\ c+90\,823\ a^6\ c-$$

$$166\,478\ a^7\ c+236\,954\ a^8\ c-269\,872\ a^9\ c+253\,478\ a^{10}\ c-208\,167\ a^{11}\ c+156\,981\ a^{12}\ c-$$

$$104\,985\ a^{13}\ c+55\,160\ a^{14}\ c-18\,760\ a^{15}\ c+2900\ a^{16}\ c-192\ a\ b\ c+1056\ a^2\ b\ c-5568\ a^3\ b\ c+$$

$$9120\ a^4\ b\ c-11\,808\ a^5\ b\ c+13\,728\ a^6\ b\ c-12\,480\ a^7\ b\ c+9024\ a^8\ b\ c-2880\ a^9\ b\ c+256\ b^2\ c+$$

$$256\ a^2\ b^2\ c-24\ a\ c^2+144\ a^2\ c^2-744\ a^3\ c^2+5568\ a^4\ c^2-13\,272\ a^5\ c^2+18\,816\ a^6\ c^2-$$

$$\left.16\,392\ a^7\ c^2+6720\ a^8\ c^2-816\ a^9\ c^2-768\ a\ b\ c^2+192\ a^2\ b\ c^2-64\ c^3+384\ a\ c^3-160\ a^2\ c^3\right)-$$

$$\frac{1}{1024\ x^{16}}9\ (-1+a)\ a\ \left(-3136\ a^3+30\,688\ a^4-150\,340\ a^5+500\,948\ a^6-1\,263\,192\ a^7+\right.$$

$$2\,529\,356\ a^8-4\,161\,776\ a^9+5\,783\,596\ a^{10}-6\,963\,353\ a^{11}+7\,472\,989\ a^{12}-7\,316\,706\ a^{13}+$$

$$6\,562\,942\ a^{14}-5\,318\,713\ a^{15}+3\,846\,233\ a^{16}-2\,501\,848\ a^{17}+1\,485\,376\ a^{18}-783\,224\ a^{19}+$$

$$332\,752\ a^{20}-96\,048\ a^{21}+13\,456\ a^{22}-2016\ a^2\ b-2400\ a^3\ b+44\,928\ a^4\ b-129\,024\ a^5\ b+$$

$$251\,232\ a^6\ b-393\,504\ a^7\ b+501\,888\ a^8\ b-536\,832\ a^9\ b+475\,392\ a^{10}\ b-368\,256\ a^{11}\ b+$$

$$264\,576\ a^{12}\ b-165\,888\ a^{13}\ b+79\,872\ a^{14}\ b-19\,968\ a^{15}\ b-3840\ a\ b^2+3328\ a^2\ b^2-6144\ a^3\ b^2+$$

$$6144\ a^4\ b^2-5632\ a^5\ b^2+6144\ a^6\ b^2-3072\ a^7\ b^2+3072\ a^8\ b^2+1776\ a^2\ c-14\,352\ a^3\ c+$$

$$84\,096\ a^4\ c-326\,784\ a^5\ c+838\,512\ a^6\ c-1\,584\,624\ a^7\ c+2\,308\,416\ a^8\ c-2\,699\,328\ a^9\ c+$$

$$2\,704\,080\ a^{10}\ c-2\,374\,080\ a^{11}\ c+1\,745\,712\ a^{12}\ c-972\,480\ a^{13}\ c+343\,104\ a^{14}\ c-54\,048\ a^{15}\ c-$$

$$6912\ a\ b\ c+40\,704\ a^2\ b\ c-71\,424\ a^3\ b\ c+110\,592\ a^4\ b\ c-132\,864\ a^5\ b\ c+121\,344\ a^6\ b\ c-$$

$$90\,624\ a^7\ b\ c+29\,184\ a^8\ b\ c-2048\ a\ b^2\ c+2304\ a\ c^2-5376\ a^2\ c^2-16\,704\ a^3\ c^2+61\,632\ a^4\ c^2-$$

$$\left.113\,280\ a^5\ c^2+112\,896\ a^6\ c^2-45\,504\ a^7\ c^2+4032\ a^8\ c^2+3072\ b\ c^2-1536\ c^3+512\ a\ c^3\right)+$$

$$\frac{1}{256\ x^{15}}9\ \left(56\ a^4+1826\ a^5-23\,218\ a^6+124\,074\ a^7-421\,139\ a^8+1\,056\,011\ a^9-2\,097\,493\ a^{10}+\right.$$

$$3\,441\,405\ a^{11}-4\,827\,044\ a^{12}+5\,935\,014\ a^{13}-6\,470\,053\ a^{14}+6\,260\,053\ a^{15}-5\,372\,693\ a^{16}+$$

$$4\,113\,699\ a^{17}-2\,830\,258\ a^{18}+1\,727\,606\ a^{19}-888\,846\ a^{20}+350\,896\ a^{21}-91\,264\ a^{22}+$$

$$11\,368\ a^{23}+3264\ a^3\ b-19\,776\ a^4\ b+57\,528\ a^5\ b-122\,136\ a^6\ b+209\,760\ a^7\ b-$$

$$295\,968\ a^8\ b+355\,560\ a^9\ b-367\,752\ a^{10}\ b+331\,104\ a^{11}\ b-266\,304\ a^{12}\ b+187\,872\ a^{13}\ b-$$

$$108\,864\ a^{14}\ b+44\,736\ a^{15}\ b-9024\ a^{16}\ b+256\ a^2\ b^2+896\ a^3\ b^2-2432\ a^4\ b^2+2688\ a^5\ b^2-$$

$$2944\ a^6\ b^2+2560\ a^7\ b^2-2048\ a^8\ b^2+1024\ a^9\ b^2+600\ a^3\ c-11\,328\ a^4\ c+67\,740\ a^5\ c-$$

$$230\,532\ a^6\ c+553\,704\ a^7\ c-1\,006\,416\ a^8\ c+1\,442\,844\ a^9\ c-1\,715\,388\ a^{10}\ c+$$

$$1\,740\,072\ a^{11}\ c-1\,485\,864\ a^{12}\ c+1\,014\,888\ a^{13}\ c-503\,712\ a^{14}\ c+154\,584\ a^{15}\ c-$$

$$21\,192\ a^{16}\ c-2688\ a^2\ b\ c+8448\ a^3\ b\ c-21\,120\ a^4\ b\ c+39\,936\ a^5\ b\ c-53\,760\ a^6\ b\ c+$$

$$54\,144\ a^7\ b\ c-33\,024\ a^8\ b\ c+8064\ a^9\ b\ c+1024\ b^2\ c+512\ a\ b^2\ c-512\ a^2\ b^2\ c+$$

$$384\,a^2 c^2 + 1632\,a^3 c^2 - 9408\,a^4 c^2 + 24\,192\,a^5 c^2 - 34\,656\,a^6 c^2 + 23\,712\,a^7 c^2 -$$
$$5664\,a^8 c^2 - 192\,a^9 c^2 - 2816\,b\,c^2 + 512\,a\,b\,c^2 + 1152\,c^3 - 384\,a\,c^3 + 128\,a^2 c^3\Big) -$$
$$\frac{1}{256\,x^{14}} 3\,\Big(7059\,a^4 - 73\,776\,a^5 + 377\,724\,a^6 - 1\,308\,618\,a^7 + 3\,461\,262\,a^8 - 7\,342\,962\,a^9 +$$
$$12\,926\,181\,a^{10} - 19\,465\,224\,a^{11} + 25\,597\,959\,a^{12} - 29\,696\,976\,a^{13} + 30\,509\,715\,a^{14} -$$
$$27\,817\,716\,a^{15} + 22\,604\,118\,a^{16} - 16\,375\,854\,a^{17} + 10\,375\,986\,a^{18} - 5\,467\,650\,a^{19} +$$
$$2\,193\,888\,a^{20} - 577\,776\,a^{21} + 72\,660\,a^{22} - 4272\,a^3 b + 29\,808\,a^4 b - 128\,160\,a^5 b + 339\,744\,a^6 b -$$
$$646\,128\,a^7 b + 970\,128\,a^8 b - 1\,164\,480\,a^9 b + 1\,163\,520\,a^{10} b - 1\,018\,368\,a^{11} b + 770\,784\,a^{12} b -$$
$$472\,320\,a^{13} b + 200\,448\,a^{14} b - 40\,704\,a^{15} b + 5376\,a\,b^2 - 2688\,a^2 b^2 - 2688\,a^3 b^2 +$$
$$4224\,a^4 b^2 - 5760\,a^5 b^2 + 5376\,a^6 b^2 - 7680\,a^7 b^2 + 3840\,a^8 b^2 - 6456\,a^3 c + 59\,544\,a^4 c -$$
$$272\,736\,a^5 c + 848\,544\,a^6 c - 1\,896\,840\,a^7 c + 3\,242\,952\,a^8 c - 4\,522\,656\,a^9 c + 5\,244\,480\,a^{10} c -$$
$$4\,957\,632\,a^{11} c + 3\,638\,136\,a^{12} c - 1\,887\,336\,a^{13} c + 591\,336\,a^{14} c - 81\,336\,a^{15} c -$$
$$18\,048\,a\,b\,c + 12\,672\,a^2 b\,c - 5376\,a^3 b\,c + 24\,192\,a^4 b\,c - 87\,936\,a^5 b\,c + 154\,752\,a^6 b\,c -$$
$$106\,752\,a^7 b\,c + 26\,496\,a^8 b\,c - 1024\,b^2 c + 2304\,a\,c^2 + 21\,600\,a^2 c^2 - 69\,984\,a^3 c^2 +$$
$$132\,192\,a^4 c^2 - 142\,848\,a^5 c^2 + 65\,232\,a^6 c^2 - 6048\,a^7 c^2 - 2448\,a^8 c^2 + 5120\,b\,c^2 - 2304\,c^3\Big) +$$
$$\frac{1}{128\,x^{13}} 3\,\Big(-3066\,a^4 + 33\,624\,a^5 - 193\,959\,a^6 + 740\,637\,a^7 - 2\,062\,356\,a^8 + 4\,508\,097\,a^9 -$$
$$8\,120\,880\,a^{10} + 12\,374\,793\,a^{11} - 16\,192\,152\,a^{12} + 18\,363\,036\,a^{13} - 18\,183\,174\,a^{14} +$$
$$15\,854\,712\,a^{15} - 12\,192\,060\,a^{16} + 8\,102\,841\,a^{17} - 4\,425\,729\,a^{18} + 1\,821\,084\,a^{19} - 487\,200\,a^{20} +$$
$$61\,752\,a^{21} - 2856\,a^2 b - 1344\,a^3 b + 7776\,a^4 b + 11\,088\,a^5 b - 109\,464\,a^6 b + 299\,040\,a^7 b -$$
$$461\,568\,a^8 b + 526\,176\,a^9 b - 514\,080\,a^{10} b + 438\,864\,a^{11} b - 302\,112\,a^{12} b + 136\,416\,a^{13} b -$$
$$27\,936\,a^{14} b - 128\,a\,b^2 + 2176\,a^2 b^2 - 1408\,a^3 b^2 - 640\,a^4 b^2 + 1024\,a^5 b^2 - 2560\,a^6 b^2 +$$
$$1536\,a^7 b^2 - 6444\,a^2 c + 47\,088\,a^3 c - 153\,792\,a^4 c + 367\,992\,a^5 c - 684\,540\,a^6 c +$$
$$1\,084\,284\,a^7 c - 1\,618\,848\,a^8 c + 2\,177\,064\,a^9 c - 2\,400\,336\,a^{10} c + 1\,979\,136\,a^{11} c -$$
$$1\,093\,644\,a^{12} c + 350\,676\,a^{13} c - 48\,636\,a^{14} c + 2816\,a\,b\,c - 3712\,a^2 b\,c - 4352\,a^3 b\,c -$$
$$5888\,a^4 b\,c + 35\,456\,a^5 b\,c - 32\,384\,a^6 b\,c + 8064\,a^7 b\,c + 1024\,b^2 c + 1344\,a\,c^2 - 14\,304\,a^2 c^2 +$$
$$47\,040\,a^3 c^2 - 67\,392\,a^4 c^2 + 41\,280\,a^5 c^2 - 6816\,a^6 c^2 - 1152\,a^7 c^2 - 2048\,b\,c^2 + 768\,c^3\Big) -$$
$$\frac{1}{256\,x^{12}} 3\,(-1 + a)\,a\,\Big(11\,088\,a^2 - 62\,508\,a^3 + 212\,172\,a^4 - 610\,116\,a^5 + 1\,519\,122\,a^6 -$$
$$3\,418\,839\,a^7 + 6\,790\,191\,a^8 - 11\,473\,770\,a^9 + 16\,275\,804\,a^{10} - 19\,314\,393\,a^{11} +$$
$$19\,499\,715\,a^{12} - 17\,168\,052\,a^{13} + 13\,151\,772\,a^{14} - 8\,382\,894\,a^{15} + 4\,054\,380\,a^{16} -$$
$$1\,269\,780\,a^{17} + 186\,108\,a^{18} + 64\,a\,b - 4608\,a^2 b + 35\,648\,a^3 b + 7360\,a^4 b - 120\,832\,a^5 b +$$
$$236\,992\,a^6 b - 347\,264\,a^7 b + 386\,560\,a^8 b - 359\,552\,a^9 b + 225\,536\,a^{10} b - 59\,904\,a^{11} b -$$
$$6144\,b^2 - 3584\,a^2 b^2 - 1280\,a^3 b^2 + 256\,a^4 b^2 - 11\,136\,a\,c + 79\,008\,a^2 c - 270\,624\,a^3 c +$$
$$495\,648\,a^4 c - 743\,424\,a^5 c + 1\,125\,264\,a^6 c - 1\,620\,384\,a^7 c + 1\,875\,504\,a^8 c - 1\,398\,768\,a^9 c +$$
$$557\,040\,a^{10} c - 88\,128\,a^{11} c + 9216\,b\,c + 16\,896\,a\,b\,c - 4096\,a^2 b\,c + 512\,a^3 b\,c +$$
$$512\,a^4 b\,c + 5376\,c^2 - 43\,200\,a\,c^2 + 56\,832\,a^2 c^2 - 38\,592\,a^3 c^2 + 10\,560\,a^4 c^2\Big) + \frac{1}{64\,x^{11}}$$
$$3\,(-1 + a)\,a\,\Big(-1077\,a^2 + 12\,015\,a^3 - 52\,404\,a^4 + 143\,055\,a^5 - 347\,646\,a^6 + 789\,300\,a^7 -$$
$$1\,590\,750\,a^8 + 2\,684\,457\,a^9 - 3\,678\,690\,a^{10} + 4\,192\,239\,a^{11} - 4\,107\,744\,a^{12} + 3\,455\,538\,a^{13} -$$
$$2\,383\,425\,a^{14} + 1\,222\,608\,a^{15} - 396\,240\,a^{16} + 58\,764\,a^{17} - 976\,a\,b + 13\,008\,a^2 b - 22\,080\,a^3 b +$$
$$27\,600\,a^4 b - 14\,160\,a^5 b - 19\,920\,a^6 b + 43\,824\,a^7 b - 57\,360\,a^8 b + 40\,928\,a^9 b - 10\,864\,a^{10} b -$$
$$1280\,a\,b^2 - 128\,a^2 b^2 - 128\,a^3 b^2 + 6984\,a\,c - 37\,560\,a^2 c + 76\,944\,a^3 c - 114\,576\,a^4 c +$$
$$148\,680\,a^5 c - 208\,440\,a^6 c + 275\,112\,a^7 c - 226\,368\,a^8 c + 95\,064\,a^9 c - 15\,840\,a^{10} c +$$
$$1536\,b\,c + 1280\,a\,b\,c + 128\,a^2 b\,c + 128\,a^3 b\,c - 3936\,c^2 + 6144\,a\,c^2 - 5472\,a^2 c^2 + 1632\,a^3 c^2\Big) -$$

$$\frac{1}{64\,x^{10}}3\left(-4788\,a^3+27\,717\,a^4-73\,035\,a^5+138\,462\,a^6-267\,096\,a^7+645\,810\,a^8-1\,546\,947\,a^9+2\,963\,988\,a^{10}-4\,497\,708\,a^{11}+5\,622\,327\,a^{12}-5\,877\,696\,a^{13}+5\,045\,019\,a^{14}-3\,366\,954\,a^{15}+1\,590\,660\,a^{16}-459\,792\,a^{17}+60\,033\,a^{18}-3328\,a^2\,b+9920\,a^3\,b-27\,008\,a^4\,b+48\,128\,a^5\,b-48\,096\,a^6\,b+22\,368\,a^7\,b+24\,384\,a^8\,b-51\,456\,a^9\,b+31\,648\,a^{10}\,b-6560\,a^{11}\,b+768\,b^2+1280\,a\,b^2-1344\,a^2\,b^2+384\,a^3\,b^2-320\,a^4\,b^2+9888\,a^2\,c-39\,168\,a^3\,c+87\,600\,a^4\,c-142\,224\,a^5\,c+211\,680\,a^6\,c-308\,400\,a^7\,c+331\,704\,a^8\,c-215\,832\,a^9\,c+76\,248\,a^{10}\,c-11\,496\,a^{11}\,c-3456\,b\,c+832\,a\,b\,c+1536\,a^2\,b\,c-384\,a^3\,b\,c+320\,a^4\,b\,c+3888\,c^2-5472\,a\,c^2+6288\,a^2\,c^2-4128\,a^3\,c^2+1008\,a^4\,c^2\right)+$$

$$\frac{1}{32\,x^9}3\left(2079\,a^3-9921\,a^4+23\,244\,a^5-32\,946\,a^6+60\,867\,a^7-187\,125\,a^8+474\,075\,a^9-920\,229\,a^{10}+1\,437\,156\,a^{11}-1\,814\,976\,a^{12}+1\,808\,772\,a^{13}-1\,341\,669\,a^{14}+675\,564\,a^{15}-201\,462\,a^{16}+26\,571\,a^{17}-1584\,a\,b+2392\,a^2\,b-6056\,a^3\,b+16\,184\,a^4\,b-31\,608\,a^5\,b+39\,600\,a^6\,b-19\,752\,a^7\,b-3800\,a^8\,b+5752\,a^9\,b-1128\,a^{10}\,b-256\,b^2-128\,a\,b^2+384\,a^2\,b^2-256\,a^3\,b^2+3564\,a\,c-6684\,a^2\,c+11\,832\,a^3\,c-21\,084\,a^4\,c+48\,012\,a^5\,c-98\,148\,a^6\,c+116\,472\,a^7\,c-77\,580\,a^8\,c+28\,248\,a^9\,c-4632\,a^{10}\,c+1728\,b\,c-384\,a^2\,b\,c+256\,a^3\,b\,c-2592\,c^2+1440\,a\,c^2-1056\,a^2\,c^2+192\,a^3\,c^2\right)+\frac{1}{64\,x^8}$$

$$\left(-9801\,a^2+15\,336\,a^3+7326\,a^4-112\,140\,a^5+280\,539\,a^6-375\,750\,a^7+310\,986\,a^8+241\,902\,a^9-1\,774\,008\,a^{10}+4\,112\,478\,a^{11}-5\,827\,635\,a^{12}+5\,280\,966\,a^{13}-2\,944\,206\,a^{14}+916\,344\,a^{15}-122\,337\,a^{16}-7200\,a\,b+6432\,a^2\,b-9984\,a^3\,b+68\,640\,a^4\,b-165\,120\,a^5\,b+163\,968\,a^6\,b-76\,512\,a^7\,b+25\,920\,a^8\,b-6144\,a^9\,b-3328\,b^2-1536\,a\,b^2+1536\,a^2\,b^2+23\,328\,a\,c-23\,184\,a^2\,c-9360\,a^3\,c-33\,120\,a^4\,c+262\,080\,a^5\,c-448\,128\,a^6\,c+342\,000\,a^7\,c-137\,808\,a^8\,c+24\,192\,a^9\,c+13\,824\,b\,c-1536\,a^2\,b\,c-15\,552\,c^2+3456\,a\,c^2-1152\,a^2\,c^2\right)+\frac{1}{16\,x^7}$$

$$\left(-2079\,a^2+2070\,a^3+14\,598\,a^4-44\,649\,a^5+72\,693\,a^6-132\,210\,a^7+239\,499\,a^8-259\,065\,a^9-3789\,a^{10}+442\,440\,a^{11}-621\,027\,a^{12}+408\,600\,a^{13}-135\,495\,a^{14}+18\,414\,a^{15}-2496\,a\,b-288\,a^2\,b+1584\,a^3\,b+8976\,a^4\,b-21\,240\,a^5\,b+25\,032\,a^6\,b-15\,288\,a^7\,b+3720\,a^8\,b-256\,b^2+5184\,a\,c+4104\,a^2\,c-19\,944\,a^3\,c+4356\,a^4\,c+34\,884\,a^5\,c-49\,896\,a^6\,c+27\,036\,a^7\,c-5724\,a^8\,c+1536\,b\,c-1728\,c^2\right)+\frac{1}{16\,x^6}$$

$$\left(-1629\,a^2-8298\,a^3+36\,738\,a^4-62\,541\,a^5+87\,417\,a^6-160\,704\,a^7+284\,427\,a^8-291\,303\,a^9+85\,284\,a^{10}+111\,888\,a^{11}-119\,997\,a^{12}+45\,054\,a^{13}-6336\,a^{14}+288\,a\,b-1056\,a^2\,b+672\,a^3\,b+2928\,a^4\,b-9216\,a^5\,b+8784\,a^6\,b-2400\,a^7\,b-256\,b^2-720\,a\,c+8496\,a^2\,c-12\,528\,a^3\,c+360\,a^4\,c+14\,112\,a^5\,c-13\,320\,a^6\,c+3600\,a^7\,c+768\,b\,c-576\,c^2\right)+$$

$$\frac{1}{8\,x^5}3\left(312\,a^2-1494\,a^3+3384\,a^4-5115\,a^5+10\,212\,a^6-23\,784\,a^7+33\,483\,a^8-23\,694\,a^9+6888\,a^{10}+156\,a^{11}-423\,a^{12}+75\,a^{13}+384\,a\,b-160\,a^2\,b+320\,a^3\,b-144\,a^4\,b-32\,a^5\,b+16\,a^6\,b+384\,c-1056\,a\,c+1008\,a^2\,c-864\,a^3\,c+216\,a^4\,c+48\,a^5\,c-24\,a^6\,c\right)-$$

$$\frac{1}{16\,x^4}3\,a\left(1344-3408\,a+3168\,a^2-1044\,a^3+1176\,a^4-13\,764\,a^5+27\,408\,a^6-20\,121\,a^7+5292\,a^8-42\,a^9-12\,a^{10}+3\,a^{11}-512\,b+256\,a\,b+1536\,c-384\,a\,c\right)+$$

$$\frac{1}{x^3}3\,a\left(24+78\,a-30\,a^2-192\,a^3-27\,a^4+549\,a^5-555\,a^6+153\,a^7+32\,b-48\,c\right)-$$

$$\frac{1}{x^2}$$

$$18\ (-1+a)^2\ a\ \left(8+6\ a-10\ a^2+a^4\right)$$

The computer proof for Step 5(a). We use maximum value as an upper bound when $-5 \le i \le -2$, and use the maximum absolute sum as an upper bound when $-20 \le i \le -6$.

In[43]:=
```
coeff = Table[If[i ≥ -5,
    Maximize[{Abs[Coefficient[G5, x^i]],
        -1 ≤ a ≤ 3 && -3994 ≤ b ≤ 64 && -1620 ≤ c ≤ 3}, {x, a, b, c}][[1]],
    FromCoefficientRules[Map[Abs, CoefficientRules[Coefficient[G5, x^i],
            {x, a, b, c}], {2}], {x, a, b, c}] /. a → 3 /. b → 3994 /. c → 1620
   ],
   {i,
    -20,
    -2}]
```

Out[43]=

$$\left\{545\,102\,954\,553\,600,\ 1\,801\,659\,993\,602\,400,\ \frac{5\,886\,126\,860\,798\,565}{2},\ 3\,236\,151\,695\,060\,880,\ \frac{19\,524\,150\,631\,163\,025}{8},\ \frac{23\,280\,405\,487\,750\,863}{16},\ 741\,299\,023\,799\,055,\ 334\,643\,780\,679\,111,\ 131\,234\,047\,150\,977,\ 44\,282\,400\,488\,163,\ 13\,051\,286\,303\,076,\ 3\,426\,985\,781\,691,\ \frac{26\,026\,504\,017\,233}{32},\ \frac{639\,099\,553\,189}{4},\ \frac{37\,583\,702\,399}{2},\ 17\,107\,740,\ 3\,629\,457,\ 1\,222\,632,\ 3672\right\}$$

The computer proof for Step 5(b).

In[44]:=
```
Simplify[Total[coeff x^Range[-20, -2]] < 1, x ≥ 120]
```

Out[44]= True

Step 6

Definition of G_6.

In[45]:=
```
G6 = Factor[μ (y - x) (mm - m0^2) /. mmsub];
Collect[G6, μ, Factor]
```

Out[46]=

$$-x^2\ (1+x)\ y\ (1+y) + \left(-x + m0^2\ x + 5\ x^2 + 4\ x^3 + 2\ y - m0^2\ y - 2\ x\ y - 2\ x^2\ y\right)\ \mu + (-1-x)\ \mu^2$$

Definition of F_4. Eq. (5.18)

In[47]:=
```
F4 = Denominator[Factor[PolynomialExtendedGCD[F3, G6, μ]][[2]]][[1]] /
    (x^2 (1 + x)^2 (x - y) y^2 (1 + y));
Collect[
 F4,
 y,
 Factor]
```

Out[48]=
$$x^5 \left(-4 + m0^2 + 4 x + 4 x^2\right) \left(-3 + m0^2 + 4 x + 4 x^2\right) \left(-3 + 3 m0^2 + 6 x + 3 m0^2 x + 21 x^2 + m0^2 x^2 + 16 x^3 + 4 x^4\right) - x^2 \left(3 m0^2 - 6 m0^4 + 3 m0^6 + 12 x - 51 m0^2 x + 36 m0^4 x + 3 m0^6 x - 204 x^2 + 204 m0^2 x^2 + 37 m0^4 x^2 + 5 m0^6 x^2 + 204 x^3 + 305 m0^2 x^3 + 60 m0^4 x^3 + 3 m0^6 x^3 + 1048 x^4 + 179 m0^2 x^4 + 80 m0^4 x^4 + m0^6 x^4 + 688 x^5 + 352 m0^2 x^5 + 48 m0^4 x^5 + 252 x^6 + 448 m0^2 x^6 + 12 m0^4 x^6 + 704 x^7 + 240 m0^2 x^7 + 832 x^8 + 48 m0^2 x^8 + 384 x^9 + 64 x^{10}\right) y + x \left(12 m0^2 - 16 m0^4 + 4 m0^6 + 48 x - 110 m0^2 x + 21 m0^4 x + 3 m0^6 x - 244 x^2 - 56 m0^2 x^2 + 34 m0^4 x^2 + 2 m0^6 x^2 - 448 x^3 + 38 m0^2 x^3 + 30 m0^4 x^3 - 180 x^4 + 8 m0^2 x^4 + 12 m0^4 x^4 - 248 x^5 - 448 x^6 - 288 x^7 - 64 x^8\right) y^2 + \left(-12 m0^2 + 7 m0^4 - m0^6 - 48 x + 20 m0^2 x - 44 x^2 + 32 m0^2 x^2 - 8 x^3 + 24 m0^2 x^3 - 44 x^4 + 12 m0^2 x^4 - 48 x^5 - 16 x^6\right) y^3$$

Step 7

Definition of $y^{(1)}$, $y^{(2)}$, $y^{(3)}$.

In[49]:=
$$y1 = -\left(2 x^3 + 3 x^2 + \frac{3}{2} m0 (m0 + 1) x + \frac{3}{4} m0 (m0 + 1)\right);$$
$$y2 = -\left(2 x^3 + 3 x^2 + \frac{3}{2} m0 (m0 - 1) x + \frac{3}{4} m0 (m0 - 1)\right);$$
```
y3 = x;
```

The computer proof for Step 7(a).

In[52]:=
```
Simplify[(F4 /. y → y1) == 0, m0 == 0]
```
Out[52]= True

The computer proof for Step 7(b).

In[53]:=
```
Simplify[(F4 /. y → y1 - 1 / 2) > 0, x ≥ 90 && 1 ≤ m0 ≤ 2]
```
Out[53]= True

The computer proof for Step 7(c).

In[54]:=
```
Simplify[(F4 /. y → y1 + 1 / 2) < 0, x ≥ 90 && 1 ≤ m0 ≤ 2]
```
Out[54]= True

The computer proof for Step 7(d).

In[55]:=
```
Simplify[(F4 /. y → y2 + 1 / x) < 0, x ≥ 90 && m0 == 0]
```
Out[55]= True

The computer proof for Step 7(e).

In[56]:=
```
Simplify[(F4 /. y → y2 - 1 / 2) < 0, x ≥ 90 && 1 ≤ m0 ≤ 2]
```
Out[56]= True

The computer proof for Step 7(f).

In[57]:=
```
Simplify[(F4 /. y → y2 + 1 / 2) > 0, x ≥ 90 && 0 ≤ m0 ≤ 2]
```
Out[57]= True

The computer proof for Step 7(g).

In[58]:=
```
Simplify[(F4 /. y → y3 - 1) > 0, x ≥ 1 && 0 ≤ m0 ≤ 2]
```
Out[58]= True

The computer proof for Step 7(h).

In[59]:=
```
Simplify[(F4 /. y → y3 + 1) < 0, x ≥ 1 && 0 ≤ m0 ≤ 2]
```
Out[59]= True

Step 8

Speed up Step 8 using multiple kernels.

In[60]:=
```
LaunchKernels[8];
```

The computer proof for Step 8.

In[61]:=
```
Select[DeleteDuplicates[Flatten[ParallelTable[
    If[(μ /. #) > 0 && y0 ≤ -1 && y0 ≠ -(2 x0^3 + 3 x0^2), {x0, y0, μ /. #}, {}] & /@
     Solve[(F3 /. x → x0 /. y → y0) == 0, μ, Integers]
    , {x0, 1, 120}, {y0, -(2 x0^3 + 3 x0^2 + 3 x0 + 2), -(2 x0^3 + 3 x0^2 - 3 x0 - 3)}], 2]],
 Length[#] > 0 &]
```
Out[61]= {{1, -1, 1}}