Explicit spherical designs

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Abstract

Since the introduction of the notion of spherical designs by Delsarte, Goethals, and Seidel in 1977, finding explicit constructions of spherical designs had been an open problem. Most existence proofs of spherical designs rely on the topology of the spheres, hence their constructive versions are only computable, but not explicit. That is to say that these constructions can only give algorithms that produce approximations of spherical designs up to arbitrary given precision, while they are not able to give any spherical designs explicitly. Inspired by recent work on rational designs, i.e. designs consisting of rational points, we generalize the known construction of spherical designs that uses interval designs with Gegenbauer weights, and give an explicit formula of spherical designs of arbitrary given strength on the real unit sphere of arbitrary given dimension.

1 Introduction

Spherical designs were introduced by Delsarte-Goethals-Seidel [9]. The first existence proof of spherical designs of arbitrary strength on arbitrary dimensional sphere was given by Seymour-Zaslavsky [16]. After that many other existence proofs were found [15, 19, 2, 13, 5, 3, 4, 8].

Once the existence of spherical designs is established, one might ask how to construct spherical designs and how explicit the constructions are. As we will recall below, there are algorithms to produce approximations of a spherical design; there also exists a very recent algorithm to produce a spherical design; the best we can hope for would be formulas that describe all the coordinates of all the points in the design, and this is what we focus on in this paper.

Most of the proofs of the existence of spherical designs use the topology of the sphere, more precisely, use the fact that the spheres are complete metric spaces. Their constructive versions then use limits of points to construct spherical designs. These constructions can only give *computable designs*, that is, designs such that there exist algorithms to compute approximations of the designs up to arbitrary given precision, or equivalently, designs with coordinates being in the field of computable real numbers $\mathbb{R}^{\mathrm{com}}$. These constructions are not explicit in the sense that they cannot give the coordinates of the points in the designs explicitly. Computable designs are good for numerical analytic purposes, and some discussion about small computable designs can be found in [12, 7, 6].

Spherical designs can be constructed from designs on simpler spaces. [15, 19, 2] gave a construction of spherical designs using designs on *Gegenbauer intervals*, that is intervals equipped with Gegenbauer weights. Although this part of the construction is explicit, it is still an open problem to give an explicit construction of designs on Gegenbauer intervals.

Recently, [8] proves the existence of *almost rational* spherical designs, that is, spherical designs where every point has rational coordinates except possibly the first coordinate. This gives the first algorithm to construct spherical designs: enumerate all finite almost rational subsets of a given sphere, and then test if they are designs or not. Since there are only countably many almost rational subsets and the existence of almost rational spherical designs is guaranteed in [8], the above procedure terminates in finite time, hence is an algorithm. For every fixed strength of the design and dimension of the sphere, designs produced by this algorithm are explicit. However, this construction is not explicit in the sense that the time required depends on the strength and the dimension. Note that designs found by this algorithm have coordinates in the field $\mathbb{Q}(\sqrt{p}\colon\ p \text{ prime})$.

On intervals (not Gegenbauer intervals), some explicit designs were constructed by Kuperberg [14]. He constructs a certain polynomial with integer coefficients and then expresses the points in the interval designs as some linear combinations of the roots of the polynomial. This gives us interval designs over the field $\mathbb{Q}^{\mathrm{alg}} \cap \mathbb{R}$, the totally real part of the algebraic closure of $\mathbb{Q}$. Note that although this construction is explicit, it is not known whether the design constructed can be written down using radical expressions or not. It is also unknown whether this approach can be generalized to spheres of dimension at least two.

In this paper, we combine the ideas of designs on Gegenbauer intervals, weighted designs and rationality of designs. Our main result is an explicit construction of spherical designs on large dimensional spheres using explicit good spherical designs on smaller dimensional spheres. In Theorem 1.2, we apply this construction to some well-known explicit good spherical designs. In particular, Theorem 1.2(i) gives explicit spherical designs of arbitrary strength on arbitrary given dimensional sphere over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$, the totally real part of the abelian closure of $\mathbb{Q}$. Theorem 1.2 is explicit in the sense that it gives a formula for each coordinate of each point in the design. Moreover, the formula can be written in finitely many symbols described below, and the number of symbols is independent of the choice of the strength and the dimension.

(i) The strength t and the dimension d;

(ii) Rational numbers, real parts of roots of unity $\operatorname{Re}\zeta_n$ and imaginary parts of roots of unity $\operatorname{Im}\zeta_n$;

(iii) Ceiling $\lceil\ \rceil$, floor $\lfloor\ \rfloor$ and arithmetic operation (sum $+$, difference $-$, product $\cdot$, quotient $/$, nonnegative square root $\sqrt{\ }$);

(iv) Finite sum $\sum_{i=a}^{b}$ and finite product $\prod_{i=a}^{b}$.

Let d be a natural number and consider the d-dimensional real unit sphere

$$S^d := \{(x_0, \dots, x_d) \in \mathbb{R}^{d+1}\colon\ x_0^2 + \cdots + x_d^2 = 1\}$$

and the d-dimensional real unit hemisphere

$$H^d := \{(x_0, \dots, x_d) \in S^d\colon\ x_0 > 0\}.$$

On the sphere S^d, let ν^d denote the spherical measure. On the hemisphere H^d, we will consider various measures ν_s^d indexed by a natural number s. The measure ν_s^d is a "shift" of the spherical measure ν^d by s, and its definition is postponed to § 2.1. We equip S^d and H^d with these measures and the resulting measure spaces are denoted by

$$\mathcal{S}^d := (S^d, \nu^d) \quad \text{and} \quad \mathcal{H}_s^d := (H^d, \nu_s^d).$$

Designs on $\mathcal{S}^d$ are subsets that approximate the sphere $\mathcal{S}^d$ nicely with respect to polynomials on $\mathcal{S}^d$. We will define in § 2.4 *semidesigns* on $\mathcal{H}_s^d$, that are subsets that approximate the hemisphere

$\mathcal{H}_s^d$ nicely with respect to exactly one half of the polynomials on $\mathcal{H}_s^d$. In § 2.4, we also generalize the *antipodal* property on $\mathcal{S}^d$ and define the *semiantipodal* property on $\mathcal{H}_s^d$.

Weighted versions of (semi)designs are used as important ingredients in this paper as well. A weighted (semi)design $\mathcal{X}$ is called *integer/rational-weighted* if all weights of $\mathcal{X}$ are integers/rationals. For a field $\mathbb{F} \subseteq \mathbb{R}$, we say that $\mathcal{X}$ is *defined over* $\mathbb{F}$, if it consists of only $\mathbb{F}$-points. In particular, $\mathcal{X}$ is called *rational* if it consists of only rational points. We call $\mathcal{X}$ *finite* provided that $\mathcal{X}$ consists of only finitely many points. The precise definitions of these concepts are postponed to § 2.2 and 2.3.

We will use designs satisfying some combinations of above properties frequently. For instance, a finite semiantipodal rational-weighted rational semidesign is a weighted semidesign such that the number of points is finite, semiantipodal property is satisfied, all weights are rational numbers, and all points are rational points.

Construction 1.1. *Let t and d be positive integers, and let $\mathcal{Y}^a$ be an explicit antipodal spherical t-design on $\mathcal{S}^a$ over a field $\mathbb{F} \subseteq \mathbb{R}$ for some positive integer $a \leq d$.*

Step 1 *Apply Corollary 3.9 to strength $t+d-1$, and get an explicit finite semiantipodal rational-weighted rational $(t+d-1)$-semidesign on $\mathcal{H}_0^1$, denoted by $\mathcal{X}_0^1$.*

Step 2 *Apply Corollary 3.12 to strength $t+d-1$, and get an explicit finite semiantipodal rational-weighted rational $(t+d-1)$-semidesign on $\mathcal{H}_1^1$, denoted by $\mathcal{X}_1^1$.*

Step 3 *For each positive even (resp. odd) integer $s < d$, apply Corollary 4.6 to $\mathcal{X}_0^1$ obtained in Step 1 (resp. $\mathcal{X}_1^1$ obtained in Step 2) and $\mathcal{H}_s^1$, and get an explicit finite semiantipodal rational-weighted rational t-semidesign $\mathcal{X}_s^1$ on $\mathcal{H}_s^1$.*

Step 4 *For each positive integer $s < d$, apply Corollary 4.10 to $\mathcal{X}_s^1, \ldots, \mathcal{X}_{d-1}^1$ obtained in Step 3, and get an explicit finite semiantipodal rational-weighted rational t-semidesign $\mathcal{X}_s^{d-s} := \mathcal{X}_s^1 \times \cdots \times \mathcal{X}_{d-1}^1$ on $\mathcal{H}_s^{d-s}$.*

Step 5 *For each positive integer $s < d$, apply Corollary 4.13 to $\mathcal{X}_s^{d-s}$ obtained in Step 4, and get an explicit semiantipodal integer-weighted rational t-semidesign $\overline{\mathcal{X}}_s^{d-s}$ on $\mathcal{H}_s^{d-s}$.*

Step 6 *Apply Corollary 4.17 to the initial design $\mathcal{Y}^a$, and get an explicit reflection $s_\alpha \in O(a+1, \mathbb{Q})$ such that $s_\alpha \mathcal{Y}^a$ is an explicit antipodal t-design on $\mathcal{S}^a$ over $\mathbb{F}$ in which every point has a nonzero first coordinate.*

Step 7 *Apply Corollary 4.25 to $s_\alpha \mathcal{Y}^a$ obtained in Step 6 and $\overline{\mathcal{X}}_a^{d-a}$ obtained in Step 5, and get an explicit antipodal t-design $\mathcal{Y}^d := (s_\alpha \mathcal{Y}^a) \rtimes_\xi \overline{\mathcal{X}}_a^{d-a}$ on $\mathcal{S}^d$ over $\mathbb{F}$, where $\rtimes_\xi$ is the* twisted product *defined in § 4.6, and ξ is a certain map $\mathbb{N} \to O(a+1, \mathbb{Q})$ determined by $s_\alpha \mathcal{Y}^a$ and defined in Corollary 4.25.*

Theorem 1.2 below is an immediate corollary of the construction applied to some well-known choices of explicit spherical designs $\mathcal{Y}^a$.

Theorem 1.2. (i) *Choose $a = 1$ and $\mathcal{Y}^1$ the set of the vertices of the regular $4(t+1)$-gon on $\mathcal{S}^1$ containing the point $(1,0)$. Then, $\mathcal{Y}^d$ is an explicit spherical t-design on $\mathcal{S}^d$ over $\mathbb{Q}(\zeta_{4(t+1)}) \cap \mathbb{R}$, where ζ_n is a primitive n-th root of unity. Moreover, all points in $\mathcal{Y}^d$ have rational coordinates except for their first two coordinates.*

(ii) *Choose $a = 3$ and $\mathcal{Y}^3$ the rational spherical 5-design from (the dual of) 24-cell on $\mathcal{S}^3$. In other words, $\mathcal{Y}^3$ consists of 24 points: 8 of them are permutations of $(\pm 1, 0, 0, 0)$, and 16 of them are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. Then, $\mathcal{Y}^d$ is an explicit rational spherical 5-design on $\mathcal{S}^d$ for every $d \geq 3$.*

(iii) *Choose $a = 7$ and $\mathcal{Y}^7$ the rational 7-design from E_8 lattice on $\mathcal{S}^7$ as in [18, 8]. Then, $\mathcal{Y}^d$ is an explicit rational spherical 7-design on $\mathcal{S}^d$ for every $d \geq 7$.*

(iv) *Choose $a = 23$ and $\mathcal{Y}^{23}$ the rational 11-design from Leech lattice on $\mathcal{S}^{23}$ as in [18, 8]. Then, $\mathcal{Y}^d$ is an explicit rational spherical 11-design on $\mathcal{S}^d$ for every $d \geq 23$.*

Since the compositum of all cyclotomic fields $\mathbb{Q}(\zeta_n)$ is $\mathbb{Q}^{\mathrm{ab}}$, the abelian closure of $\mathbb{Q}$, we can construct t-designs over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$ for all t, as shown in the following corollary.

Corollary 1.3. *For every positive integers t and d, Theorem 1.2(i) gives an explicit t-design on $\mathcal{S}^d$ over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$.*

Except for the initial design $\mathcal{Y}^a$, we only uses rational objects to construct the final design $\mathcal{Y}^d$. Therefore, field automorphisms commute with the construction. This observation gives the following immediate corollary.

Corollary 1.4. *Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield, and let $\sigma \in \mathrm{Aut}(\mathbb{F} / \mathbb{Q})$ be a field automorphism. Let $\mathcal{Y}^a$ and $\mathcal{Y}^d$ be as in the construction. If $\mathcal{Y}^a$ is stable under the action of σ, then $\mathcal{Y}^d$ is also stable under the action of σ.*

Eiichi Bannai first asked the question whether rational spherical designs exist or not. Theorem 1.2(ii), (iii) and (iv) give explicit affirmative answers when $t \leq 5$ and $d \geq 3$, $t \leq 7$ and $d \geq 7$, or $t \leq 11$ and $d \geq 23$. It motivates us to give Conjecture 1.5.

Conjecture 1.5. For every positive integer t, there exists a rational spherical t-design on $\mathcal{S}^d$ for some positive integer d.

Note that if Conjecture 1.5 is true, applying the construction, we can get a rational spherical t-design on $\mathcal{S}^d$ for all sufficiently large d.

The paper is organized as follows. We introduce necessary concepts in § 2. In § 3, we consider one-dimensional semicircle $\mathcal{H}^1_s$, and do *Step 1* and *Step 2*. In § 4, we consider high-dimensional hemispheres $\mathcal{H}^d_s$ and spheres $\mathcal{S}^d$, and do *Step 3* to *Step 7*. We will explain at the beginning of subsections of § 4 the motivations of the corresponding steps.

Notation. Throughout the paper, for a real interval I, the set of all integers in I is denoted by $I_{\mathbb{Z}}$. For instance, $[0, t)_{\mathbb{Z}}$ consists of all natural numbers smaller than t, and $[1, d]_{\mathbb{Z}}$ consists of all positive integers no greater than d.

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2 Preliminary

We use measure theory [17] as the foundation of design theory.

2.1 Radon-Nikodym derivative

Let X be a *measurable space*, that is a set equipped with a σ-algebra consisting of subsets of X called *measurable sets*. The *Radon-Nikodym derivative* of a measure μ on X with respect to another measure ν on X, denoted by $\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}$, is a measurable function $X \to \mathbb{R}^{\geq 0}$ such that

$$\mu(E) = \int_E \frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}\,\mathrm{d}\,\nu \tag{2.1}$$

for all measurable sets E. In probability theoretic language, the Radon-Nikodym derivative is known as the probability density function.

When we know the measure ν, by specifying the Radon-Nikodym derivative $\frac{\mathrm{d}\,\mu}{\mathrm{d}\,\nu}$, Eq. (2.1) gives a description of the measure μ. We use this method to construct certain measures ν_s^d on the hemisphere H^d indexed by a natural number s.

Definition 2.1. Let s be a natural number and $g_s(x_0, \ldots, x_d) := x_0^s$, regarded as a polynomial on H^d. Let ν_s^d be the measure on H^d with Radon-Nikodym derivative with respect to the spherical measure ν^d

$$\frac{\mathrm{d}\,\nu_s^d}{\mathrm{d}\,\nu^d} = g_s.$$

The topological space H^d equipped with the measure ν_s^d is denoted by $\mathcal{H}_s^d$.

It might be worth explaining here why we use hemispheres in our approach instead of intervals and balls.

Let I be the real open interval $(-1, 1)$, and let $\mathcal{I}_s$ be the interval I equipped with the density function $x \mapsto (1 - x^2)^{s/2}$. The interval $\mathcal{I}_s$ is sometimes called a *Gegenbauer interval*, or an *interval with Gegenbauer weight*. Consider the natural projection to the second coordinate $H^1 \to I$. This projection induces an isomorphism of measure spaces $\mathcal{H}_s^1 \xrightarrow{\sim} \mathcal{I}_{s-1}$. In high dimensional space, we have a similar isomorphism of measure spaces $\mathcal{H}_s^d \xrightarrow{\sim} \mathcal{B}_{s-1}^d$, where $\mathcal{B}_{s-1}^d$ is the d-dimensional real open unit ball equipped with a certain measure indexed by $s - 1$.

However, under this projection, a rational point on I might not lift to a rational point on H^1. In the construction, for the final design $\mathcal{Y}^d$ being defined over the same field $\mathbb{F}$ as the initial design $\mathcal{Y}^a$, we found that it is necessary to work with rational points on hemispheres, and not with rational points on intervals or balls. Using hemispheres also allows us to only handle polynomials and eliminate the use of radical expressions like $\sqrt{1 - x^2}$ in most parts of the construction.

2.2 Levelling spaces and related constructions

In this paper, we use the notation in [8]. For readers' convenience, we repeat some important definitions here.

A *levelling space* $\mathcal{X} = (X, \mu_X)$ is a nonempty Hausdorff topological space X, which is called the *support*, equipped with a measure μ_X on X such that the total measure is finite and the measures of nonempty open sets are positive.

We use the convention that $\mathcal{X} \subseteq \mathcal{Z} = (Z, \mu_Z)$ means only that X is a topological subspace of Z, and we assume nothing on the measures μ_X and μ_Z.

The *total measure* of $\mathcal{X}$ is denoted by $|\,\mathcal{X}\,|$, and it is clear that $|\,\mathcal{X}\,| = \int_X 1\,\mathrm{d}\,\mu_X$. We say that a levelling space $\mathcal{X}$ is *finite* if its cardinality is finite, namely, $\mathcal{X}$ consists of finitely many points.

We say that $\mathcal{X}$ is *rational-weighted* (resp. *integer-weighted*) if the image of μ_X is contained in $\mathbb{Q}$ (resp. $\mathbb{Z}$). When X is finite, the counting measure on X is denoted by $\underline{1}_X$, namely $\underline{1}_X(E) = |E|$ for all subsets $E \subseteq X$. We say that $\mathcal{X}$ is $\underline{1}$-*weighted* if the measure μ_X is the counting measure $\underline{1}_X$.

A *map* between levelling spaces $\mathcal{X}$ and $\mathcal{Y}$ is a map that is both a continuous map of topological spaces and a measurable map of measure spaces. A map ι is a *dominant open embedding* provided that ι is a dominant open embedding of topological spaces (i.e. ι maps the domain homeomorphically to the image, and the image is dense in the codomain) and the measure of the image equals to the measure of the codomain. If we have a dominant open embedding between $\mathcal{X}$ and $\mathcal{Y}$, then we can basically think of them as the same for our purposes, as to be shown in Lemma 4.3.

Given two levelling space $\mathcal{X}$ and $\mathcal{Y}$, and a map $\iota : \mathcal{X} \to \mathcal{Y}$, we can define some related levelling spaces as follows.

(i) *Scalar* $c\,\mathcal{X} := (X, c\mu_X)$ for positive real c. For every continuous measurable function f on $\mathcal{X}$,

$$\int_X f \,\mathrm{d}\, c\mu_X = c \int_X f \,\mathrm{d}\, \mu_X. \tag{2.2}$$

(ii) *Sum* $\mathcal{X} + \mathcal{Y} := (X \cup Y, \mu_X + \mu_Y)$. Note that we only have this sum if the topologies of X and Y agree on the set intersection $X \cap Y$. For every continuous measurable function f on $\mathcal{X} + \mathcal{Y}$,

$$\int_{X\cup Y} f \,\mathrm{d}(\mu_X + \mu_Y) = \left(\int_X f \,\mathrm{d}\, \mu_X\right) + \left(\int_Y f \,\mathrm{d}\, \mu_Y\right). \tag{2.3}$$

(iii) *Product* $\mathcal{X} \times \mathcal{Y} := (X \times Y, \mu_X \times \mu_Y)$. For every continuous measurable function f on $\mathcal{X}$ and g on $\mathcal{Y}$,

$$\int_{X\times Y} f \otimes g \,\mathrm{d}\, \mu_X \times \mu_Y = \left(\int_X f \,\mathrm{d}\, \mu_X\right)\left(\int_Y g \,\mathrm{d}\, \mu_Y\right). \tag{2.4}$$

(iv) *Image* $\iota(\mathcal{X}) := (\iota(X), \iota_*\mu_X)$, where $\iota_*\mu_X$ is the pushforward measure of μ_X along the map ι. For every continuous measurable function f on $\iota(\mathcal{X})$,

$$\int_X \iota^* f \,\mathrm{d}\, \mu_X = \int_{\iota(X)} f \,\mathrm{d}\, \iota_*\mu_X, \quad \text{where} \quad \iota^* f := f \circ \iota. \tag{2.5}$$

2.3 Designs and polynomials

Let $\mathcal{Z} = (Z, \mu_Z)$ be a levelling space and V a real vector space of continuous integrable functions on $\mathcal{Z}$. A V*-design* on $\mathcal{Z}$ is a levelling space $\mathcal{X} = (X, \mu_X) \subseteq \mathcal{Z}$ such that

$$\frac{1}{|\,\mathcal{X}\,|} \int_X f \,\mathrm{d}\, \mu_X = \frac{1}{|\,\mathcal{Z}\,|} \int_Z f \,\mathrm{d}\, \mu_Z \tag{2.6}$$

for all $f \in V$. For a field $\mathbb{F} \subseteq \mathbb{R}$, we call $\mathcal{X}$ a V*-design over* $\mathbb{F}$ when it makes sense to talk about $\mathbb{F}$-points in $\mathcal{Z}$ and $\mathcal{X}$ consists of only $\mathbb{F}$-points in $\mathcal{Z}$. The V-designs over $\mathbb{Q}$ are also called *rational* V*-designs.*

Let $Z \subseteq \mathbb{R}^{d+1}$ be a topological subspace. Denote $\mathcal{P}^t[Z]$ the vector space of all polynomials on Z with degree bounded above by t. We use the convention that $\mathcal{P}^\infty[Z]$ is the vector space of all polynomials on Z. It is clear that

$$\mathcal{P}^t[S^d] \cong \mathcal{P}^t[H^d] \cong \mathbb{R}[x_0, \ldots, x_d]_{\le t}/(x_0^2 + \cdots + x_d^2 - 1)_{\le t},$$

where subscript $\le t$ means the degree $\le t$ part.

Definition 2.2. Assume that polynomials are integrable on $\mathcal{Z} = (Z, \mu_Z)$. A *weighted* t*-design on* $\mathcal{Z}$ is a $\mathcal{P}^t[Z]$-design on $\mathcal{Z}$. A t*-design on* $\mathcal{Z}$ is a $\underline{1}$-weighted $\mathcal{P}^t[Z]$-design on $\mathcal{Z}$.

In particular, a (weighted) t-design on $\mathcal{S}^d$ is just an ordinary (weighted) spherical t-design on $\mathcal{S}^d$.

2.4 Semidesigns and antipodal maps

For tuples $\mathbf{x} = (x_0, \ldots, x_d)$ and $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_d)$, let $\mathbf{x}^{\boldsymbol{\lambda}} := \prod_{i=0}^{d} x_i^{\lambda_i}$. For each natural number r, let

$$\mathcal{P}^{t,r}[\mathbb{R}^{d+1}] := \mathbb{R}\langle \mathbf{x}^{\boldsymbol{\lambda}} :\ \boldsymbol{\lambda} \in \mathbb{N}^{d+1},\ \textstyle\sum_{i=0}^{r} \lambda_i \text{ is even}\rangle_{\le t},$$

and set $\mathcal{P}^{t,0,r}[\mathbb{R}^{d+1}] := \mathcal{P}^{t,0}[\mathbb{R}^{d+1}] \cap \mathcal{P}^{t,r}[\mathbb{R}^{d+1}]$.

Since the defining equation of both S^d and H^d, $x_0^2 + \cdots + x_d^2 - 1$, is in $\mathcal{P}^{t,r}[\mathbb{R}^{d+1}]$ for all r, we have the following well-defined quotients:

$$\begin{aligned} \mathcal{P}^{t,0}[H^d] &:= \mathcal{P}^{t,0}[\mathbb{R}^{d+1}]/(x_0^2 + \cdots + x_d^2 - 1)_{\le t} \subseteq \mathcal{P}^t[H^d]; \\ \mathcal{P}^{t,0,r}[H^d] &:= \mathcal{P}^{t,0,r}[\mathbb{R}^{d+1}]/(x_0^2 + \cdots + x_d^2 - 1)_{\le t} \subseteq \mathcal{P}^t[H^d]; \\ \mathcal{P}^{t,r}[S^d] &:= \mathcal{P}^{t,r}[\mathbb{R}^{d+1}]/(x_0^2 + \cdots + x_d^2 - 1)_{\le t} \subseteq \mathcal{P}^t[S^d]. \end{aligned} \tag{2.7}$$

It is easy to see that the vector space of polynomials on H^d admits a direct sum decomposition

$$\mathcal{P}^t[H^d] = \mathcal{P}^{t,0}[H^d] \oplus x_0\, \mathcal{P}^{t-1,0}[H^d].$$

Definition 2.3. A levelling space $\mathcal{X} \subseteq H^d$ is a *weighted t-semidesign* on $\mathcal{H}_s^d$ provided that $\mathcal{X}$ is a $\mathcal{P}^{t,0}[H^d]$-design on $\mathcal{H}_s^d$. A weighted t-semidesign is called a *t-semidesign* if it is $\underline{1}$-weighted.

The name *semidesign* comes from the fact that $\mathcal{P}^{\infty}[H^d] \cong \mathcal{P}^{\infty,0}[H^d] \oplus \mathcal{P}^{\infty,0}[H^d]$.

Definition 2.4. Let a, b be natural numbers such that $a \le b \le d$. The *$[a,b]$-antipodal map* is defined to be

$$-_{[a,b]} : (x_0, \ldots, x_d) \mapsto (x_0, \ldots, x_{a-1}, -x_a, \ldots, -x_b, x_{b+1}, \ldots, x_d).$$

A set $X \subseteq \mathbb{R}^{d+1}$ (resp. levelling space $\mathcal{X} \subseteq \mathbb{R}^{d+1}$) is called *$[a,b]$-antipodal* if $-_{[a,b]}X = X$ (resp. $-_{[a,b]}\,\mathcal{X} = \mathcal{X}$). We call a set/levelling space *antipodal* (resp. *semiantipodal*) if it is $[0,d]$-antipodal (resp. $[1,d]$-antipodal).

Note that $-_{[a,b]}\,\mathcal{X}$ should be understood as the image of $\mathcal{X}$ under the map $-_{[a,b]}$, and it follows from $-_{[a,b]}\,\mathcal{X} = \mathcal{X}$ that $\mu_X(-_{[a,b]}E) = \mu_X(E)$ for all measurable sets E.

Lemma 2.5. *Let $\mathcal{X} \subseteq H^d$ be a $[1,r]$-antipodal levelling space. Then, $\mathcal{X}$ is a weighted t-semidesign on $\mathcal{H}_s^d$ if and only if $\mathcal{X}$ is a $\mathcal{P}^{t,0,r}[H^d]$-design on $\mathcal{H}_s^d$.*

Proof. Let $\mathbf{x}^{\boldsymbol{\lambda}} \in \mathcal{P}^{t,0}[H^d]$ be a monomial. When $\sum_{i=1}^{r} \lambda_i$ is even, $\mathbf{x}^{\boldsymbol{\lambda}} \in \mathcal{P}^{t,0,r}[H^d]$. When $\sum_{i=1}^{r} \lambda_i$ is odd, since both $\mathcal{H}_s^d$ and $\mathcal{X}$ are $[1,r]$-antipodal,

$$\frac{1}{|\,\mathcal{H}_s^d\,|} \int_{H^d} \mathbf{x}^{\boldsymbol{\lambda}} \,\mathrm{d}\,\nu_s^d = \frac{1}{|\,\mathcal{X}\,|} \int_X \mathbf{x}^{\boldsymbol{\lambda}} \,\mathrm{d}\,\mu_X = 0. \qquad \square$$

Lemma 2.6. *Let $\mathcal{X} \subseteq S^d$ be a $[0,r]$-antipodal levelling space. Then, $\mathcal{X}$ is a weighted t-design on $\mathcal{S}^d$ if and only if $\mathcal{X}$ is a $\mathcal{P}^{t,r}[S^d]$-design on $\mathcal{S}^d$.*

Proof. The result can be proved by using similar arguments as in the proof of Lemma 2.5. $\square$

3 Designs on semicircles

In § 3.1, we show another interpretation of semidesigns on the semicircle $\mathcal{H}_s^1$. In § 3.2, with the help of this interpretation, we give a strategy to construct explicit weighted semidesigns on the semicircle $\mathcal{H}_s^1$. In § 3.3 and § 3.4, we apply this strategy to $\mathcal{H}_0^1$ and $\mathcal{H}_1^1$ and do *Step 1* and *Step 2*, respectively.

We always use $(t-1)$-semidesigns in this section, since most formulas would look more complicated if we used t-semidesigns.

3.1 Vandermonde matrix and semidesigns

For every finite subset $\mathbf{a}$ of the interval $(-1, 1)$, we associate it a finite subset $X_{\mathbf{a}}$ of the semicircle H^1 as follows:

$$X_{\mathbf{a}} := \{x_a \in H^1 : \; a \in \mathbf{a}\} \quad \text{where} \quad x_a := (\sqrt{1-a^2}, a) \in H^1 \,. \tag{3.1}$$

We will show in Lemma 3.3 that semidesigns on $\mathcal{H}^1_s$ with support $X_{\mathbf{a}}$ can be viewed as positive solutions $\mathbf{x}$ of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}_s$, where $\mathbf{A}$ is the Vandermonde matrix of $\mathbf{a}$ defined in Definition 3.1, and $\mathbf{b}_s$ is some column vector determined by $\mathcal{H}^1_s$ and defined in Lemma 3.2. Some estimate on the behavior of $\mathbf{A}$ is given in Lemma 3.5.

Definition 3.1. Let $\mathbf{a}$ be a finite set of real numbers. The *t-th Vandermonde matrix* of $\mathbf{a}$ is the matrix $\mathbf{A}$ with rows indexed by $[0, t)_{\mathbb{Z}}$, columns indexed by $\mathbf{a}$ and entries $\mathbf{A}_{i,a} := a^i$ for $i \in [0, t)_{\mathbb{Z}}$ and $a \in \mathbf{a}$. The *Vandermonde matrix* of $\mathbf{a}$ is the $|\mathbf{a}|$-th Vandermonde matrix of $\mathbf{a}$, which is an invertible square matrix.

Lemma 3.2. *Let s and i be natural numbers. Let*

$$b_{s,i} := \frac{1}{|\,\mathcal{H}^1_s\,|} \int_{H^1} y_1^i \,\mathrm{d}\,\nu^1_s, \tag{3.2}$$

where $y_1^i \in \mathbb{R}[y_1] \cong \mathcal{P}^{\infty,0}[H^1]$ is regarded as a polynomial on $\mathcal{H}^1_s$. Then,

$$b_{s,i} = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ \dfrac{\Gamma(\frac{s+2}{2})\Gamma(\frac{i+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{s+i+2}{2})} \in \mathbb{Q}, & \text{if } i \text{ is even,} \end{cases}$$

where Γ is the gamma function. In particular,

$$b_{0,i} = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ \frac{1}{2^i}\binom{i}{\frac{i}{2}}, & \text{if } i \text{ is even,} \end{cases} \quad \text{and} \quad b_{1,i} = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ \frac{1}{i+1}, & \text{if } i \text{ is even.} \end{cases} \tag{3.3}$$

Proof. It follows from [10]. □

Lemma 3.3. *Let s, t, n be natural numbers, and let $\mathbf{a}$ be a finite subset of the interval $(-1, 1)$ of size n and $\mathbf{A}$ the t-th Vandermonde matrix of $\mathbf{a}$. We associate every measure χ on $X_{\mathbf{a}}$ a column vector $\mathbf{x} = (\chi(x_a) : \; a \in \mathbf{a})$, and vice versa. Then, $(X_{\mathbf{a}}, \chi)$ is a weighted $(t-1)$-semidesign on $\mathcal{H}^1_s$ with $\chi(X_{\mathbf{a}}) = n$ if and only if $\mathbf{A}\mathbf{x} = n\mathbf{b}_s$ and $\mathbf{x}$ is positive, where $\mathbf{b}_s := (b_{s,i} : \; i \in [0, t)_{\mathbb{Z}})$ and $b_{s,i}$ is defined in Eq. (3.2).*

Proof. The result follows from rewriting the definition of semidesigns, particularly Eq. (2.6), in terms of Vandermonde matrix $\mathbf{A}$ and constants $b_{s,i}$. □

The L^∞-norm of a (column) vector $\mathbf{a}$ is the maximum absolute value of coordinates of $\mathbf{a}$:

$$\|\mathbf{a}\|_\infty := \max_{i \in I} |a_i|,$$

where I is the index set for rows. It induces an L^∞-norm on a matrix $\mathbf{A}$, which is the maximum absolute row sum of $\mathbf{A}$:

$$\|\mathbf{A}\|_\infty := \max_{i \in I} \sum_{a \in \mathbf{a}} |\mathbf{A}_{i,a}|, \tag{3.4}$$

where $\mathbf{a}$ is the index set for columns.

Definition 3.4. Let X and $\widetilde{X}$ be subsets of a metric space. Fix a bijection $\sim\,:\, X \to \widetilde{X}$.

(i) Let $\operatorname{dist}_{\min} X$ be the infimum of distances among distinct points of X, namely

$$\operatorname{dist}_{\min} X := \inf\{\operatorname{dist}(x, y) :\ x, y \in X, x \neq y\}.$$

(ii) Let $\operatorname{dist}(\widetilde{X}, X)$ be the distance between $\widetilde{X}$ and X with respect to $\sim$, namely

$$\operatorname{dist}(\widetilde{X}, X) := \sup_{x \in X} \operatorname{dist}(\widetilde{x}, x).$$

We analyze in Lemma 3.5 the norms of Vandermonde matrices and related matrices.

Lemma 3.5. *Let* $\mathbf{a}$ *(resp.* $\widetilde{\mathbf{a}}$*) be a subset of the interval* $(-1, 1)$ *of size* t *and* $\mathbf{A}$ *(resp.* $\widetilde{\mathbf{A}}$*) the Vandermonde matrix of* $\mathbf{a}$ *(resp.* $\widetilde{\mathbf{a}}$*). Fix a bijection* $\sim\, :\, \mathbf{a} \to \widetilde{\mathbf{a}}$*. Then, the following statements hold.*

(i) *Let* $\delta := \operatorname{dist}_{\min} \mathbf{a}$*. Then,*

$$\|\mathbf{A}\|_\infty = t \quad \textit{and} \quad \|\mathbf{A}^{-1}\|_\infty \leq (2/\delta)^{t-1}.$$

(ii) *Let* $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$*. Then,*

$$\|\widetilde{\mathbf{A}} - \mathbf{A}\|_\infty \leq t(t-1)\varepsilon.$$

Proof. (i) Since $\mathbf{a}$ consists of numbers in $(-1, 1)$, it is straightforward to calculate $\|\mathbf{A}\|_\infty$, and [11] gives the desired estimate of $\|\mathbf{A}^{-1}\|_\infty$.
(ii) We expand $\|\widetilde{\mathbf{A}} - \mathbf{A}\|_\infty$ according to Eq. (3.4) and estimate it.

$$\begin{aligned}
&\|\widetilde{\mathbf{A}} - \mathbf{A}\|_\infty \\
=& \max_{i \in [0,t)_{\mathbb{Z}}} \sum_{a \in \mathbf{a}} |\widetilde{a}^i - a^i| \qquad \text{(by Eq. (3.4) and Definition 3.1)} \\
=& \max_{i \in [0,t)_{\mathbb{Z}}} \sum_{a \in \mathbf{a}} \left(|\widetilde{a} - a| \cdot \left| \sum_{k=0}^{i-1} \widetilde{a}^k a^{i-k-1} \right| \right) \\
\leq& \max_{i \in [0,t)_{\mathbb{Z}}} \sum_{a \in \mathbf{a}} \varepsilon i \qquad \text{(since } |\widetilde{a} - a| \leq \varepsilon \text{ and } \widetilde{a}, a \in (-1,1)) \\
=& t(t-1)\varepsilon. \qquad \text{(since } |\mathbf{a}| = t) \quad \square
\end{aligned} \tag{3.5}$$

3.2 A strategy to construct designs on $\mathcal{H}_s^1$

Here is our strategy to construct weighted semidesigns on $\mathcal{H}_s^1$. First, we start with some subset X of H^1 which is "almost" a semidesign when every point has measure 1. Then, we choose a subset $\widetilde{X}$ of H^1 to approximate X. At the end, we choose a subset $\widetilde{X}'$ of $\widetilde{X}$, and tweak the measure on $\widetilde{X}'$ to get a weighted semidesign. Theorem 3.6 shows the details of this strategy.

Theorem 3.6. *Let* s, t, n *be natural numbers such that* $2 \leq t \leq n$*. Let* $\mathbf{a}$ *and* $\widetilde{\mathbf{a}}$ *be subsets of the interval* $(-1, 1)$ *of size* n *and fix a bijection* $\sim\, :\, \mathbf{a} \to \widetilde{\mathbf{a}}$*. Let* $\mathbf{a}'$ *be a subset of* $\mathbf{a}$ *of size* t*, and let* $\widetilde{\mathbf{a}}'$

be the image of $\mathbf{a}'$ *in* $\widetilde{\mathbf{a}}$ *under the bijection. In other words, we assume that we have the following commutative diagram.*

$$\begin{tikzcd} \mathbf{a} \text{ of size } n \arrow[r, "\sim"] & \widetilde{\mathbf{a}} \text{ of size } n \\ \mathbf{a}' \text{ of size } t \arrow[r, "\sim"] \arrow[u, hook] & \widetilde{\mathbf{a}}' \text{ of size } t \arrow[u, hook] \end{tikzcd} \tag{3.6}$$

Let

$$\boldsymbol{\epsilon} := (\boldsymbol{\epsilon}_i : \ i \in [0,t)_{\mathbb{Z}}), \quad \text{where} \quad \boldsymbol{\epsilon}_i := \frac{1}{n} \sum_{a \in \mathbf{a}} a^i - b_{s,i}, \tag{3.7}$$

and

$$\widetilde{\boldsymbol{\epsilon}} := (\widetilde{\boldsymbol{\epsilon}}_i : \ i \in [0,t)_{\mathbb{Z}}), \quad \text{where} \quad \widetilde{\boldsymbol{\epsilon}}_i := \frac{1}{n} \sum_{\widetilde{a} \in \widetilde{\mathbf{a}}} \widetilde{a}^i - b_{s,i}, \tag{3.8}$$

where $b_{s,i}$ *is some rational number defined in Lemma 3.2. Let* $\delta' := \operatorname{dist}_{\min} \mathbf{a}'$ *(see Definition 3.4(i)) and* $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$ *(see Definition 3.4(ii)). Assume that*

$$t^2 \varepsilon + n \|\boldsymbol{\epsilon}\|_\infty < (\delta'/2)^{t-1}. \tag{3.9}$$

Then, $X_{\widetilde{\mathbf{a}}}$ *(see Eq. (3.1)) is the support of a unique weighted* $(t-1)$*-semidesign* $\mathcal{X} = (X_{\widetilde{\mathbf{a}}}, \chi)$ *on* $\mathcal{H}_s^1$ *such that* $\chi(X_{\widetilde{\mathbf{a}}}) = n$ *and* $\chi(x) = 1$ *for every* $x \in X_{\widetilde{\mathbf{a}}} \setminus X_{\widetilde{\mathbf{a}}'}$*. Moreover, the unique measure* χ *is given by*

$$\chi(x) = \begin{cases} 1 - n \displaystyle\sum_{i=0}^{t-1} (-1)^i \frac{e_{t-i-1}(\widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\})}{\prod_{\widetilde{b}' \in \widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\}} (\widetilde{b}' - \widetilde{a}')} \widetilde{\boldsymbol{\epsilon}}_i, & \text{if } x = x_{\widetilde{a}'} \in X_{\widetilde{\mathbf{a}}'}, \\ 1, & \text{if } x \in X_{\widetilde{\mathbf{a}}} \setminus X_{\widetilde{\mathbf{a}}'}, \end{cases} \tag{3.10}$$

where $e_{t-i-1}(\widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\})$ *is the* $(t-i-1)$*-th elementary polynomial in* $t-1$ *numbers* $\widetilde{\mathbf{a}}' \setminus \{\widetilde{a}'\}$*, and we adopt the convention that* $e_0 = 1$*.*

Proof. Let $\mathbf{A}$, $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}'$ be the t-th Vandermonde matrix of $\mathbf{a}$, $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{a}}'$, respectively. The definitions of $\boldsymbol{\epsilon}$ and $\widetilde{\boldsymbol{\epsilon}}$, Eqs. (3.7) and (3.8), are equivalent to

$$\mathbf{A}\mathbf{1} = n\mathbf{b}_s + n\boldsymbol{\epsilon} \quad \text{and} \quad \widetilde{\mathbf{A}}\mathbf{1} = n\mathbf{b}_s + n\widetilde{\boldsymbol{\epsilon}}. \tag{3.11}$$

Let $\widetilde{\mathbf{x}}' \in \mathbb{R}^t$ be an indeterminate vector and $\widetilde{\mathbf{x}} \in \mathbb{R}^n$ the extension of $\widetilde{\mathbf{x}}'$ by 0 according to the inclusion $\widetilde{\mathbf{a}}' \subseteq \widetilde{\mathbf{a}}$. Consider the equation

$$\widetilde{\mathbf{A}}(\mathbf{1} + \widetilde{\mathbf{x}}) = n\mathbf{b}_s. \tag{3.12}$$

By the definition of $\widetilde{\mathbf{x}}$, we have $\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} = \widetilde{\mathbf{A}}'\widetilde{\mathbf{x}}'$, hence Eq. (3.12) is equivalent to

$$\widetilde{\mathbf{A}}\mathbf{1} + \widetilde{\mathbf{A}}'\widetilde{\mathbf{x}}' = n\mathbf{b}_s. \tag{3.13}$$

Since $\widetilde{\mathbf{A}}'$ is an invertible square Vandermonde matrix, using Eq. (3.11), we see that Eq. (3.13) has a unique solution

$$\widetilde{\mathbf{x}}' = -n\widetilde{\mathbf{A}}'^{-1}\widetilde{\boldsymbol{\epsilon}}. \tag{3.14}$$

Lemma 3.3 and the constraints of the desired design implies that $X_{\widetilde{\mathbf{a}}}$ is the support of a desired design if and only if Eq. (3.12) has a solution $\widetilde{\mathbf{x}}$ such that $\mathbf{1} + \widetilde{\mathbf{x}} > \mathbf{0}$. Moreover, the associated measure χ is uniquely determined by $\mathbf{1} + \widetilde{\mathbf{x}}$, and Eq. (3.10) is the expansion of $\mathbf{1} + \widetilde{\mathbf{x}}$ using Eq. (3.14) and the explicit formula for the inverse of the Vandermonde matrix $\widetilde{\mathbf{A}}'$.

Let $\widetilde{\delta}' := \operatorname{dist}_{\min} \widetilde{\mathbf{a}}'$. Clearly,

$$\widetilde{\delta}' \geq \delta' - 2\varepsilon. \tag{3.15}$$

Now, we estimate the L^∞-norm of $\widetilde{\mathbf{x}}'$.

$$\begin{aligned}
\|\widetilde{\mathbf{x}}'\|_\infty =& \|n\widetilde{\mathbf{A}}'^{-1}\widetilde{\boldsymbol{\epsilon}}\|_\infty && \text{(by Eq. (3.14))}\\
=& \left\|\widetilde{\mathbf{A}}'^{-1}\left((\widetilde{\mathbf{A}}-\mathbf{A})\mathbf{1}+n\boldsymbol{\epsilon}\right)\right\|_\infty && \text{(by Eq. (3.11))}\\
\leq& \|\widetilde{\mathbf{A}}'^{-1}\|_\infty\left(\|\widetilde{\mathbf{A}}-\mathbf{A}\|_\infty\cdot\|\mathbf{1}\|_\infty+n\|\boldsymbol{\epsilon}\|_\infty\right) && \text{(by properties of norms)}\\
\leq& \frac{t(t-1)\varepsilon\cdot 1+n\|\boldsymbol{\epsilon}\|_\infty}{(\widetilde{\delta}'/2)^{t-1}} && \text{(by Lemma 3.5)}\\
\leq& \frac{t(t-1)\varepsilon\cdot 1+n\|\boldsymbol{\epsilon}\|_\infty}{(\delta'/2-\varepsilon)^{t-1}} && \text{(by ?? 3.15)}\\
\leq& \frac{t(t-1)\varepsilon\cdot 1+n\|\boldsymbol{\epsilon}\|_\infty}{(\delta'/2)^{t-1}-(t-1)\varepsilon} && \text{(since } 2\varepsilon<\delta'<2 \text{ by ?? 3.9)}\\
<& 1. && \text{(by ?? 3.9)}
\end{aligned}$$

Therefore, $\mathbf{1}+\widetilde{\mathbf{x}} > \mathbf{0}$, and the result follows from Lemma 3.3. ☐

Remark 3.7. It is clear that for the degree d elementary symmetric polynomial in $a_1, \ldots, a_n$, we have

$$e_d(a_1, \ldots, a_n) = \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \sum_{j_d=j_{d-1}+1}^{n} \prod_{k=1}^{d} a_{j_k}.$$

However, this expression uses d finite sums $\sum$. With some encoding/decoding techniques, it is possible to write down $e_d(a_1, \ldots, a_n)$ using only a constant number of finite sum $\sum$, finite product $\prod$ and other operations mentioned in the introduction.

3.3 *Step 1*: Designs on $\mathcal{H}_0^1$

In § 3.3, we fix a positive integer $t \geq 2$, and let $n := t$. Let $\mathbf{a} := \{a_j : j \in [0, n)_{\mathbb{Z}}\}$ where

$$a_j := \operatorname{Im} \zeta_{4n}^{-n+1+2j} = \sin \frac{-n+1+2j}{n}\frac{\pi}{2}. \tag{3.16}$$

The set $X_{\mathbf{a}}$ (see Eq. (3.1)) is semiantipodal (see Definition 2.4) since $a_j = -a_{n-j-1}$. For each $j \in [0, n)_{\mathbb{Z}}$, we choose some to-be-determined approximation $\widetilde{a}_j$ of a_j so that $x_{\widetilde{a}_j}$ (see Eq. (3.1)) is a rational point on H^1 and $\widetilde{a}_j = -\widetilde{a}_{n-j-1}$. We set $\widetilde{\mathbf{a}} := \{\widetilde{a}_j : j \in [0, n)_{\mathbb{Z}}\}$, and the set $X_{\widetilde{\mathbf{a}}}$ is also semiantipodal. We fix a bijection $\sim\, : \mathbf{a} \to \widetilde{\mathbf{a}}$ where $a_j \mapsto \widetilde{a}_j$. Choose $\mathbf{a}' := \mathbf{a}$ and $\widetilde{\mathbf{a}}' := \widetilde{\mathbf{a}}$.

Then, $\mathbf{a}$, $\widetilde{\mathbf{a}}$, $\mathbf{a}'$ and $\widetilde{\mathbf{a}}'$ form the commutative diagram in Eq. (3.6), all of $X_{\mathbf{a}}$, $X_{\widetilde{\mathbf{a}}}$, $X_{\mathbf{a}'}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal, and $X_{\widetilde{\mathbf{a}}}$ and $X_{\widetilde{\mathbf{a}}'}$ are rational.

Lemma 3.8. *Let $\boldsymbol{\epsilon}$ be defined using $\mathbf{a}$ (see Eq. (3.16)) and $b_{0,j}$ (see Eq. (3.3)) as in Eq. (3.7). Then,*

$$\|\boldsymbol{\epsilon}\|_\infty = 0.$$

Proof. It is well-known that the vertices of a regular $2t$-gon give a spherical $(2t-1)$-design on $\mathcal{S}^1$, hence a $(t-1)$-design on $\mathcal{S}^1$. The set $X_{\mathbf{a}}$ (see Eq. (3.1)) is one half of the vertices of a regular $2t$-gon and $\mathcal{H}_0^1$ is one half of $\mathcal{S}^1$. By symmetry, $X_{\mathbf{a}}$, when equipped with the counting measure, is a $(t-1)$-semidesign. The result follows immediately from Lemma 3.3. ☐

Corollary 3.9. *Let* $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$ *(see Definition 3.4(ii)). If*

$$\varepsilon \le \frac{\pi^{2t-5}}{2^{t-4}t^{2t}}, \tag{3.17}$$

then, applying Theorem 3.6 to $\mathbf{a}$, $\widetilde{\mathbf{a}}$, $\mathbf{a}'$ *and* $\widetilde{\mathbf{a}}'$, *we get an explicit finite semiantipodal rational-weighted rational* $(t-1)$*-semidesign on* $\mathcal{H}_0^1$.

Proof. Let $\delta' := \operatorname{dist}_{\min} \mathbf{a}'$ (see Definition 3.4(i)). Eq. (3.16) gives

$$\delta' = a'_{t-1} - a'_{t-2} = a_{t-1} - a_{t-2} = 2\sin\frac{\pi}{2t}\sin\frac{\pi}{t},$$

and Lemma 3.8 gives $\|\boldsymbol{\epsilon}\|_\infty = 0$. For $z \in [0, \frac{\pi}{2}]$,

$$\sin z \ge z e^{-cz}, \quad \text{where } c := \left(\log\frac{\pi}{2}\right)/\frac{\pi}{2}. \tag{3.18}$$

Then,

$$\begin{aligned} t^{-2}\left((\delta'/2)^{t-1} - n\|\boldsymbol{\epsilon}\|_\infty\right) =& t^{-2}\left(\sin\frac{\pi}{2t}\sin\frac{\pi}{t}\right)^{t-1} \\ \ge& t^{-2}\left(\frac{\pi}{2t}\exp\left(-c\frac{\pi}{2t}\right)\frac{\pi}{t}\exp\left(-c\frac{\pi}{t}\right)\right)^{t-1} && \text{(by Eq. (3.18))} \\ =& \frac{\pi^{2t-5}}{2^{t-4}t^{2t}}\left(\frac{\pi}{2}\right)^{3/t} \\ >& \varepsilon, && \text{(by ?? 3.17)} \end{aligned}$$

from which ?? 3.9 follows. Therefore, Theorem 3.6 gives an explicit weighted $(t-1)$-semidesign $\mathcal{X} = (X_{\widetilde{\mathbf{a}}}, \chi)$.

Recall from Definition 2.4 that the $[1,1]$-antipodal map is denoted by $-_{[1,1]}$, and $\mathcal{X}$ is semiantipodal if $-_{[1,1]}\mathcal{X} = \mathcal{X}$. Since both $X_{\widetilde{\mathbf{a}}}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal, the levelling space $\frac{1}{2}\left(-_{[1,1]}\mathcal{X}\right) + \frac{1}{2}\mathcal{X}$ is also a $(t-1)$-semidesign that satisfies conditions $\chi(X_{\widetilde{\mathbf{a}}}) = n$ and $\chi(x) = 1$ for $x \in X_{\widetilde{\mathbf{a}}} \setminus X_{\widetilde{\mathbf{a}}'}$. According to the uniqueness of $\mathcal{X}$ showed by Theorem 3.6, $\mathcal{X} = \frac{1}{2}\left(-_{[1,1]}\mathcal{X}\right) + \frac{1}{2}\mathcal{X}$, hence $\mathcal{X}$ is semiantipodal.

Since every $x_{\widetilde{a}_i}$ is a rational point on H^1, $\mathcal{X}$ is a rational semidesign. Since every $\widetilde{a}_i$ is a rational number and every $b_{0,j}$ is a rational number (see Eq. (3.3)), every $\widetilde{\boldsymbol{\epsilon}}_j$ is then a rational number (see Eq. (3.8)). Therefore, the explicit formula for χ (see Eq. (3.10)) shows that $\mathcal{X}$ is rational-weighted. The semidesign $\mathcal{X}$ is finite since its support $X_{\widetilde{\mathbf{a}}}$ is finite. ☐

Remark 3.10. One explicit choice for $\widetilde{a}_j$ with $j < (n-1)/2$ is

$$\widetilde{a}_j := \frac{2\widetilde{c}_j}{\widetilde{c}_j^2 + 1} \quad \text{where} \quad \widetilde{c}_j := \frac{\lfloor m c_j \rfloor}{m}, \quad c_j := \frac{1 - \sqrt{1 - a_j^2}}{a_j}, \quad \text{and} \quad m := 2t^{2t}.$$

The other $\widetilde{a}_j$'s are obtained from $\widetilde{a}_j = -\widetilde{a}_{n-j-1}$. Let $f(z) := \frac{2z}{z^2+1}$. Clearly, $f(c_j) = a_j$ and $f(\widetilde{c}_j) = \widetilde{a}_j$. Since $|\frac{\mathrm{d}f}{\mathrm{d}z}| \le 2$,

$$\varepsilon \le \max_j(a_j - \widetilde{a}_j) \le 2\max_j(c_j - \widetilde{c}_j) \le \frac{2}{m} = \frac{1}{t^{2t}} \le \frac{\pi^{2t-5}}{2^{t-4}t^{2t}}.$$

3.4 *Step 2*: Designs on $\mathcal{H}_1^1$

The goal of § 3.4 is to construct a certain $(t-1)$-semidesign. When t is odd, we can first construct a t-semidesign with desired properties, and then regard it as a $(t-1)$-semidesign. Therefore, we can assume that t is even.

In § 3.4, we fix a positive even integer t and fix a to-be-determined positive even integer n such that $n \geq t \geq 2$ and n/t is an odd integer. Let $\mathbf{a} := \{a_j : j \in [0,n)_{\mathbb{Z}}\}$ where

$$a_j := \frac{-n+1+2j}{n}. \tag{3.19}$$

The set $X_{\mathbf{a}}$ (see Eq. (3.1)) is semiantipodal (see Definition 2.4) since $a_j = -a_{n-j-1}$. For each $j \in [0,n)_{\mathbb{Z}}$, we choose some to-be-determined approximation $\widetilde{a}_j$ of a_j so that $x_{\widetilde{a}_j}$ (see Eq. (3.1)) is a rational point on H^1 and $\widetilde{a}_j = -\widetilde{a}_{n-j-1}$. We set $\widetilde{\mathbf{a}} := \{\widetilde{a}_j : j \in [0,n)_{\mathbb{Z}}\}$, and the set $X_{\widetilde{\mathbf{a}}}$ is also semiantipodal. We fix a bijection $\sim\, : \mathbf{a} \to \widetilde{\mathbf{a}}$ where $a_j \mapsto \widetilde{a}_j$.

For each $j \in [0,t)_{\mathbb{Z}}$, let

$$a'_j := a_{\ell(j)} = \frac{-t+1+2j}{t}, \quad \text{where} \quad \ell(j) := \frac{(2j+1)n-t}{2t} \in [0,n)_{\mathbb{Z}}, \tag{3.20}$$

and set $\mathbf{a}' := \{a'_j : j \in [0,t)_{\mathbb{Z}}\}$. Let $\widetilde{\mathbf{a}}'$ be the image of $\mathbf{a}$ under the bijection $\sim$. Since $a'_j = -a'_{t-j-1}$ and $\widetilde{a}'_j = -\widetilde{a}'_{t-j-1}$, both $X_{\mathbf{a}'}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal.

Then, as in case $\mathcal{H}_0^1$, we also get the commutative diagram Eq. (3.6), all of $X_{\mathbf{a}}$, $X_{\widetilde{\mathbf{a}}}$, $X_{\mathbf{a}'}$ and $X_{\widetilde{\mathbf{a}}'}$ are semiantipodal, and $X_{\widetilde{\mathbf{a}}}$ and $X_{\widetilde{\mathbf{a}}'}$ are rational.

Lemma 3.11. *Let $\boldsymbol{\epsilon}$ be defined using $\mathbf{a}$ (see Eq. (3.19)) and $b_{1,i}$ (see Eq. (3.3)) as in Eq. (3.7). Then,*

$$\|\boldsymbol{\epsilon}\|_\infty < \frac{t}{4n^2}.$$

Proof. When i is odd, $\frac{1}{n}\sum_{j=0}^{n-1} a_j^i = 0$ since $X_{\mathbf{a}}$ is semiantipodal, and Eq. (3.3) gives $b_{1,i} = 0$, hence $\boldsymbol{\epsilon}_i = \frac{1}{n}\sum_{j=0}^{n-1} a_j^i - b_{1,i} = 0 < \frac{t}{4n^2}$. From now on, assume that i is even.

The sum of powers of first n integers can be calculated using Bernoulli's formula:

$$\sum_{j=1}^{n} j^i = \frac{1}{i+1}\sum_{k=0}^{i}(-1)^k \binom{i+1}{k} B_k n^{i+1-k}. \tag{3.21}$$

Applying this formula to a_i's, we get

$$\begin{aligned}
\boldsymbol{\epsilon}_i =& \frac{2}{n}\sum_{j=n/2}^{n-1} a_j^i - \frac{1}{i+1} && \text{(by Eq. (3.3), and } a_j = -a_{n-1-j}) \\
=& \frac{2}{n^{i+1}}\left(\sum_{j=1}^{n} j^i - 2^i \sum_{j=1}^{n/2} j^i\right) - \frac{1}{i+1} && \text{(by Eq. (3.19))} \\
=& \frac{1}{i+1}\sum_{k=0}^{i}\binom{i+1}{k}(-1)^k B_k (2-2^k) n^{-k} - \frac{1}{i+1} && \text{(by Eq. (3.21))} \\
=& -\frac{1}{i+1}\sum_{\text{even } k=2}^{i}\binom{i+1}{k} B_k (2^k-2) n^{-k}.
\end{aligned}$$

(since $B_0 = 1$, $B_k = 0$ for odd $k \geq 3$)

For positive even k, [1, 23.1.15] gives a bound on Bernoulli number:

$$|B_k| < \frac{2\,k!}{(2\pi)^k}\frac{1}{1-2^{1-k}}. \tag{3.22}$$

We use this bound to estimate $\boldsymbol{\epsilon}_i$.

$$\begin{aligned}
|\boldsymbol{\epsilon}_i| \leq & \frac{1}{i+1}\sum_{\text{even k=2}}^{i}\binom{i+1}{k}|B_k|(2^k-2)n^{-k} \\
< & \frac{1}{i+1}\sum_{\text{even } k=2}^{\infty}\frac{(i+1)^k}{k!}\frac{2\,k!}{(2\pi)^k}\frac{1}{1-2^{1-k}}(2^k-2)n^{-k} && \text{(by Eq. (3.22))} \\
= & \frac{2}{i+1}\sum_{\text{even } k=2}^{\infty}\left(\frac{i+1}{\pi n}\right)^k = \frac{2(i+1)}{(\pi n)^2-(i+1)^2} \\
\leq & \frac{2(i+1)}{(\pi n)^2 - n^2} < \frac{i+1}{4n^2}. && \text{(since } i+1 \leq t \leq n\text{)}
\end{aligned}$$

Therefore, $\|\boldsymbol{\epsilon}\|_\infty = \min_{i\in[0,t)_{\mathbb{Z}}}|\boldsymbol{\epsilon}_i| < \frac{t}{4n^2}$. □

Corollary 3.12. *Let* $\varepsilon := \operatorname{dist}(\widetilde{\mathbf{a}}, \mathbf{a})$ *(see Definition 3.4(ii)). If*

$$\varepsilon < \frac{1}{t^{t+1}} - \frac{1}{4nt}, \tag{3.23}$$

then, applying Theorem 3.6 to $\mathbf{a}$, $\widetilde{\mathbf{a}}$, $\mathbf{a}'$ *and* $\widetilde{\mathbf{a}}'$, *we get an explicit finite semiantipodal rational-weighted rational* $(t-1)$*-semidesign on* $\mathcal{H}_1^1$.

Proof. We follow the strategy as in the proof of Corollary 3.9. Let $\delta' := \operatorname{dist}_{\min}\mathbf{a}'$ (see Definition 3.4(i)). Eq. (3.20) gives

$$\delta' = a'_{t-1} - a'_{t-2} = \frac{2}{t},$$

?? 3.23 gives an upper bound on ε and Lemma 3.11 gives an upper bound for $\|\boldsymbol{\epsilon}\|_\infty$. The result follows from similar arguments to Corollary 3.9. □

Remark 3.13. One explicit choice for n is $n := (t^{t-1}+1)t$. For this particular choice of n, one explicit choice for $\widetilde{a}_j$ with $j < (n-1)/2$ in Corollary 3.12 is

$$\widetilde{a}_j := \frac{2\widetilde{c}_j}{\widetilde{c}_j^2+1} \quad \text{where} \quad \widetilde{c}_j := \frac{\lfloor mc_j \rfloor}{m}, \quad c_j := \frac{1-\sqrt{1-a_j^2}}{a_j} \quad \text{and} \quad m := 3t^{t+1}.$$

The other $\widetilde{a}_j$'s are obtained from $\widetilde{a}_j = -\widetilde{a}_{n-j-1}$. We can prove similarly as in 3.10 that

$$\epsilon \leq \frac{2}{m} < \frac{1}{t^{t+1}} - \frac{1}{4nt}.$$

4 Designs on spheres and hemispheres

§ 4.1 studies the structure of spheres and hemispheres. Proposition 4.2 allows us to view spheres and hemispheres as products of lower dimensional spheres and hemispheres. In each of § 4.2, § 4.3, § 4.4, § 4.5, and § 4.6, we show a different type of construction of designs in general spaces and explain why we need it in the final construction. Then, we specialize it to spheres and hemispheres and do *Step 3*, *Step 4*, *Step 5*, *Step 6* and *Step 7*, respectively.

4.1 Structure of spheres and hemispheres

Let a and b be two natural numbers. Consider the double branched cover of topological spaces

$$\begin{array}{cccccc} \iota_{a,b}: & S^a & \times & S^b & \to & S^{a+b}, \\ & (x_0,\dots,x_a) & \times & (y_0,\dots,y_b) & \mapsto & (x_0y_0,\dots,x_ay_0,y_1,\dots,y_b). \end{array}$$

The map $\iota_{a,b}$ induces a dominant open embedding (see § 2.2) of topological spaces

$$\iota_{a,b}\colon\ S^a\times H^b\to S^{a+b} \tag{4.1}$$

and an isomorphism of topological spaces

$$\iota_{a,b}\colon\ H^a\times H^b\to H^{a+b}\,. \tag{4.2}$$

Remark 4.1. The map $\iota_{a,b}$, regarded as a product operator, is associative in the sense that we have the following commutative diagram.

$$\begin{tikzcd} H^a\times H^b\times H^c \arrow[r, "\iota_{a,b}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\iota_{b,c}"] & H^{a+b}\times H^c \arrow[d, "\iota_{a+b,c}"] \\ H^a\times H^{b+c} \arrow[r, "\iota_{a,b+c}"] & H^{a+b+c} \end{tikzcd}$$

By associativity, $\iota_{a,b}$ induces an isomorphism of topological spaces

$$\iota_{(1^d)}\colon\ \underbrace{H^1\times\cdots\times H^1}_{d\text{ copies of }H^1}\to H^d, \tag{4.3}$$

which can be explicitly described as

$$((x_{i,0},x_{i,1})\colon\ i\in[1,d]_{\mathbb{Z}})\mapsto\left(x_{i,1}\prod_{j=i+1}^{d}x_{j,0}\colon\ i\in[0,d]_{\mathbb{Z}}\right),\quad\text{where } x_{0,1}:=x_{1,0}.$$

Recall that in § 2.2, we define maps and dominant open embeddings of levelling spaces. When we equip the spheres and hemispheres with suitable measures, we can make the maps in Eqs. (4.1) to (4.3) dominant open embedding or isomorphisms of levelling spaces.

Proposition 4.2. *Let a,b,s be natural numbers. The following statements hold.*

(i) *The map $\iota_{a,b}$ in Eq. (4.1) induces a dominant open embedding of levelling spaces*

$$\iota_{a,b}\colon\ \mathcal{S}^a\times\mathcal{H}^b_a\to\mathcal{S}^{a+b}\,.$$

(ii) *The map $\iota_{a,b}$ in Eq. (4.2) induces an isomorphism of levelling spaces*

$$\iota_{a,b}\colon\ \mathcal{H}^a_s\times\mathcal{H}^b_{a+s}\to\mathcal{H}^{a+b}_s\,.$$

(iii) *The map $\iota_{(1^d)}$ in Eq. (4.3) induces an isomorphism of levelling spaces*

$$\iota_{1^d}\colon\ \mathcal{H}^1_s\times\cdots\times\mathcal{H}^1_{s+d-1}\to\mathcal{H}^d_s\,.$$

Proof. (ii) Using the parametrization $(x_1, \dots, x_d) \mapsto (x_0, \dots, x_d)$ of the hypersurface $\mathcal{H}_s^d$, for all continuous measurable function f on $\mathcal{H}_s^d$,

$$\int_{H^d} f \,\mathrm{d}\,\nu_s^d = \int_{B^d} x_0^{s-1} f(x_0, \dots, x_d) \,\mathrm{d}\,x_1 \dots \mathrm{d}\,x_d, \tag{4.4}$$

where $B^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d :\ x_1^2 + \cdots + x_d^2 < 1\}$ is the d-dimensional unit open ball. With Eq. (4.4) and separation of variables, it is easy to check that for all continuous integrable function f on $\mathcal{H}_s^{a+b}$,

$$\int_{H^{a+b}} f \,\mathrm{d}\,\nu_s^{a+b} = \int_{H^b} \left(\int_{H^a} \iota_{a,b}^* f \,\mathrm{d}\,\nu_s^a \right) \mathrm{d}\,\nu_{a+s}^b, \quad \text{where} \quad \iota_{a,b}^* :\ f \mapsto f \circ \iota_{a,b},$$

from which the result follows.
(i): This can be proved similarly as in (ii).
(iii): Since $\iota_{a,b}$, regarded as a product operator, is associative, the result follows from an induction on (ii). □

Recall that the image $\iota(\mathcal{X}) = (\iota(X), \iota_* \mu_X)$ of a levelling space $\mathcal{X}$ under a map ι is defined in § 2.2.

Lemma 4.3. *Suppose that $\iota :\ \mathcal{W} \to \mathcal{Z}$ is a dominant open embedding of levelling spaces. Let V be a real vector space of continuous integrable functions on $\mathcal{Z}$. Let $\iota^* V := \{\iota^* f :\ f \in V\}$. Then, for every $\iota^* V$-design $\mathcal{X}$ on $\mathcal{W}$, $\iota(\mathcal{X})$ is a V-design on $\mathcal{Z}$.*

Proof. For arbitrary $f \in V$,

$$\begin{aligned}
&\frac{1}{|\iota(\mathcal{X})|} \int_{\iota(X)} f \,\mathrm{d}\,\iota_* \mu_X \\
=&\frac{1}{|\,\mathcal{X}\,|} \int_X \iota^* f \,\mathrm{d}\,\mu_X && \text{(by Eq. (2.5))} \\
=&\frac{1}{|\,\mathcal{W}\,|} \int_W \iota^* f \,\mathrm{d}\,\mu_W && \text{(since } \iota^* f \in \iota^* V,\ \mathcal{X} \text{ is an } \iota^* V\text{-design on } \mathcal{W}) \\
=&\frac{1}{|\iota(\mathcal{W})|} \int_{\iota(W)} f \,\mathrm{d}\,\iota_* \mu_W && \text{(by Eq. (2.5))} \\
=&\frac{1}{|\,\mathcal{Z}\,|} \int_Z f \,\mathrm{d}\,\mu_Z. && \text{(since } \iota \text{ is a dominant open embedding)}
\end{aligned}$$

Therefore, $\iota(\mathcal{X})$ is a V-design on $\mathcal{Z}$. □

Remark 4.4. In the remaining sections, we regard $\mathcal{S}^a \times \mathcal{H}_a^b$ as a subspace of $\mathcal{S}^{a+b}$ using the dominant open embedding $\mathcal{S}^a \times \mathcal{H}_a^b \to \mathcal{S}^{a+b}$ in Proposition 4.2(i), and regard designs on $\mathcal{S}^a \times \mathcal{H}_a^b$ as designs on $\mathcal{S}^{a+b}$ using Lemma 4.3 without explicitly mentioning it. We do similar identifications for the dominant open embeddings $\mathcal{H}_s^a \times \mathcal{H}_{a+s}^b \to \mathcal{H}_s^{a+b}$ in Proposition 4.2(ii) and $\mathcal{H}_s^1 \times \cdots \times \mathcal{H}_{s+d-1}^1 \to \mathcal{H}_a^d$ in Proposition 4.2(iii).

4.2 *Step 3*: Lifts of designs

The goal of *Step 3* is to reduce the need of constructing designs on $\mathcal{H}_s^1$ for all positive integers $s < d$ down to constructing designs for only finitely many s independent with the choice of d. For our purpose, we only need construct designs manually for $s = 0$, which is done in *Step 1*, and for $s = 1$, which is done in *Step 2*.

Recall that the Radon-Nikodym derivative is defined in § 2.1.

Lemma 4.5. *Let Z be a measurable space, $\mathcal{Z}_0 = (Z, \mu_{Z_0})$ and $\mathcal{Z}_1 = (Z, \mu_{Z_1})$ two levelling spaces on Z, and $\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}$ the Radon-Nikodym derivative of μ_{Z_1} with respect to μ_{Z_0}. Let V_0 (resp. V_1) be a real vector space of continuous integrable functions on $\mathcal{Z}_0$ (resp. $\mathcal{Z}_1$). Assume that*

$$V_0 \supseteq \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} V_1 := \left\{ \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} f : \ f \in V_1 \right\}.$$

Let $\mathcal{X}_0 = (X, \mu_{X_0})$ be a V_0-design on $\mathcal{Z}_0$. Then, $\mathcal{X}_1 := (X, \mu_{X_1})$ with

$$\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} = \frac{\mathrm{d}\,\mu_{X_1}}{\mathrm{d}\,\mu_{X_0}} \tag{4.5}$$

is a V_1-design on $\mathcal{Z}_1$.

Proof. Let f be an arbitrary function in V_1. Then,

$$\begin{aligned}
&\frac{1}{|\,\mathcal{Z}_1\,|} \int_Z f \,\mathrm{d}\,\mu_{Z_1} \\
=& \left(\int_Z f \,\mathrm{d}\,\mu_{Z_1} \right) / \left(\int_Z 1 \,\mathrm{d}\,\mu_{Z_1} \right) \\
=& \left(\int_Z f \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} \,\mathrm{d}\,\mu_{Z_0} \right) / \left(\int_Z 1 \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} \,\mathrm{d}\,\mu_{Z_0} \right) && \text{(by Eq. (2.1))} \\
=& \left(\int_X f \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} \,\mathrm{d}\,\mu_{X_0} \right) / \left(\int_X 1 \frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} \,\mathrm{d}\,\mu_{X_0} \right) \\
& && \left(\text{since } f\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}}, 1\frac{\mathrm{d}\,\mu_{Z_1}}{\mathrm{d}\,\mu_{Z_0}} \in V_0, \text{ and } \mathcal{X}_0 \text{ is a } V_0\text{-design}\right) \\
=& \left(\int_X f \frac{\mathrm{d}\,\mu_{X_1}}{\mathrm{d}\,\mu_{X_0}} \,\mathrm{d}\,\mu_{X_0} \right) / \left(\int_X 1 \frac{\mathrm{d}\,\mu_{X_1}}{\mathrm{d}\,\mu_{X_0}} \,\mathrm{d}\,\mu_{X_0} \right) && \text{(by Eq. (4.5))} \\
=& \left(\int_X f \,\mathrm{d}\,\mu_{X_1} \right) / \left(\int_X 1 \,\mathrm{d}\,\mu_{X_1} \right) && \text{(by Eq. (2.1))} \\
=& \frac{1}{|\,\mathcal{X}_1\,|} \int_X f \,\mathrm{d}\,\mu_{X_1},
\end{aligned}$$

from which the result follows. □

Corollary 4.6. *Let s and $\widetilde{s}$ be two natural numbers such that $s - \widetilde{s}$ is an even natural number. Let $\mathcal{X}^1_{\widetilde{s}} = (X, \mu^1_{\widetilde{s}})$ be a finite semiantipodal rational-weighted rational $(t+s-\widetilde{s})$-semidesign on $\mathcal{H}^1_{\widetilde{s}}$. Then, $\mathcal{X}^1_s := (X, \mu^1_s)$, where*

$$\mu^1_s(x_0, x_1) := x_0^{s-\widetilde{s}} \mu^1_{\widetilde{s}}(x_0, x_1). \tag{4.6}$$

is a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}^1_s$,

Proof. By Definition 2.1, the Radon-Nikodym derivative $\mathrm{d}\,\nu^1_s / \mathrm{d}\,\nu^1_{\widetilde{s}}$ is $g_{s-\widetilde{s}}$, which is a degree $s - \widetilde{s}$ polynomial in $\mathcal{P}^{\infty,0}[H^1]$ since $s - \widetilde{s}$ is an even natural number. Eq. (4.6) shows that the Radon-Nikodym derivative $\mathrm{d}\,\mu^1_s / \mathrm{d}\,\mu^1_{\widetilde{s}}$ is $g_{s-\widetilde{s}}$ as well. Since

$$g_{s-\widetilde{s}} \mathcal{P}^{t,0}[H^1] \subseteq \mathcal{P}^{t+s-\widetilde{s},0}[H^1],$$

by Lemma 4.5, $\mathcal{X}^1_s$ is a weighted t-semidesign on $\mathcal{H}^1_s$. The semiantipodal and finite property of $\mathcal{X}^1_s$ inherits from the semiantipodal and finite property of $\mathcal{X}^1_{\widetilde{s}}$, and the rationality of $\mathcal{X}^1_s$ follows from the rationality of $\mathcal{X}^1_{\widetilde{s}}$ and Eq. (4.6). □

Remark 4.7. It is straightforward to generalize Corollary 4.6 to higher dimensional hemispheres $\mathcal{H}^d_s$ and $\mathcal{H}^d_{\widetilde{s}}$.

4.3 *Step 4*: Products of designs

The goal of *Step 4* is to construct designs on high dimensional hemispheres using known designs on semicircles which we constructed in *Step 3*.

Recall that the product of levelling spaces is defined in § 2.2.

Lemma 4.8. *For each $i \in \{0,1\}$, let $\mathcal{Z}_i$ be a levelling space, V_i a real vector space of continuous integrable functions on $\mathcal{Z}_i$, and $\mathcal{X}_i$ a V_i-design on $\mathcal{Z}_i$. Then, $\mathcal{X}_0 \times \mathcal{X}_1$ is a $V_0 \otimes V_1$-design on $\mathcal{Z}_0 \times \mathcal{Z}_1$.*

Proof. For arbitrary $f_0 \in V_0$ and $f_1 \in V_1$,

$$\begin{aligned}
&\frac{1}{|\,\mathcal{X}_0 \times \mathcal{X}_1\,|} \int_{X_0 \times X_1} f_0 \otimes f_1 \,\mathrm{d}\,\mu_{X_0} \times \mu_{X_1} \\
=&\left(\frac{1}{|\,\mathcal{X}_0\,|} \int_{X_0} f_0 \,\mathrm{d}\,\mu_{X_0}\right)\left(\frac{1}{|\,\mathcal{X}_1\,|} \int_{X_1} f_1 \,\mathrm{d}\,\mu_{X_1}\right) && \text{(by Eq. (2.4))} \\
=&\left(\frac{1}{|\,\mathcal{Z}_0\,|} \int_{Z_0} f_0 \,\mathrm{d}\,\mu_{Z_0}\right)\left(\frac{1}{|\,\mathcal{Z}_1\,|} \int_{Z_1} f_1 \,\mathrm{d}\,\mu_{Z_1}\right) \\
& && \text{(since } f_i \in V_i,\ \mathcal{X}_i \text{ is a } V_i\text{-design on } \mathcal{Z}_i) \\
=&\frac{1}{|\,\mathcal{Z}_0 \times \mathcal{Z}_1\,|} \int_{Z_0 \times Z_1} f_0 \otimes f_1 \,\mathrm{d}\,\mu_{Z_0} \times \mu_{Z_1}. && \text{(by Eq. (2.4))}
\end{aligned}$$

Since $V_0 \otimes V_1$ is generated by functions of the form $f_0 \otimes f_1$, $\mathcal{X}_0 \times \mathcal{X}_1$ is a $V_0 \otimes V_1$-design. □

Recall from Definition 2.4 that we call a levelling space $\mathcal{X}$ semiantipodal if it is stable under a certain semiantipodal map.

Lemma 4.9. *Let $\mathcal{X}_s^a$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^a$ and let $\mathcal{X}_{a+s}^b$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_{a+s}^b$. Then, $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$, regarded as a subspace of $\mathcal{H}_s^{a+b}$ using Remark 4.4, is a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^{a+b}$.*

Proof. We label the coordinates of H^a, H^b and H^{a+b} as follows

$$H^a = \{(x_0, \dots, x_a)\}, \quad H^b = \{(y_0, \dots, y_b)\}, \quad H^{a+b} = \{(z_0, \dots, z_{a+b})\}.$$

Since both $\mathcal{X}_s^a$ and $\mathcal{X}_{a+s}^b$ are semiantipodal, the product $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$, regarded as a subspace of $\mathcal{H}_s^{a+b}$ by Remark 4.4, is both semiantipodal and $[1,a]$-antipodal.

Recall that $\mathcal{P}^{t,0}$ and $\mathcal{P}^{t,0,a}$ are defined in Eq. (2.7), and $\mathcal{P}^{t,0,a}$ is a subfunctor of $\mathcal{P}^{t,0}$. Consider the dominant open embedding $\iota_{a,b}$ in Proposition 4.2(ii). Its comorphism, the pullback $\iota_{a,b}^*$, gives an inclusion

$$\begin{aligned}
\iota_{a,b}^* : \mathcal{P}^{t,0,a}[H^{a+b}] &\hookrightarrow \mathcal{P}^{t,0}[H^a] \otimes \mathcal{P}^{t,0}[H^b], \\
z_i &\mapsto \begin{cases} x_i \otimes y_0, & \text{if } i \in [1,a]_{\mathbb{Z}}, \\ 1 \otimes y_{i-a}, & \text{if } i \in [a+1, a+b]_{\mathbb{Z}}. \end{cases}
\end{aligned}$$

Note that although y_0 is not in $\mathcal{P}^{t,0}[H^b]$, any monomial in $\mathcal{P}^{t,0,a}[H^{a+b}]$ maps to some monomial with even degree in y_0, hence in $\mathcal{P}^{t,0}[H^b]$.

Since $\mathcal{X}_s^a$ is a $\mathcal{P}^{t,0}[H^a]$-design on $\mathcal{H}_s^a$ and $\mathcal{X}_{a+s}^b$ is a $\mathcal{P}^{t,0}[H^b]$-design on $\mathcal{H}_{a+s}^b$, according to Lemma 4.8, $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$ is a $\mathcal{P}^{t,0}[H^a] \otimes \mathcal{P}^{t,0}[H^b]$-design on $\mathcal{H}_s^a \times \mathcal{H}_{a+s}^b$, hence a $\mathcal{P}^{t,0,a}[H^{a+b}]$-design on $\mathcal{H}_s^a \times \mathcal{H}_{a+s}^b$ by the comorphism $\iota_{a,b}^*$. According to Lemma 2.5, the $[1,a]$-antipodal $\mathcal{P}^{t,0,a}[H^{a+b}]$-design $\mathcal{X}_s^a \times \mathcal{X}_{a+s}^b$ is a weighted t-semidesign on $\mathcal{H}_s^{a+b}$.

Since both of $\mathcal{X}_s^a$ and $\mathcal{X}_{a+s}^b$ are finite, rational-weighted and rational, their product is finite, rational-weighted and rational as well. □

Corollary 4.10. *For each $i \in [s,d)_{\mathbb{Z}}$, let $\mathcal{X}_i^1$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_i^1$. Then, $\mathcal{X}_s^1 \times \cdots \times \mathcal{X}_{d-1}^1$, regarded as a subspace of $\mathcal{H}_s^{d-s}$ using Remark 4.4, is a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^{d-s}$.*

Proof. The result follows from Proposition 4.2(iii) and induction using Lemma 4.9. □

4.4 *Step 5*: Weights of designs

The goal of *Step 5* is to view a finite rational-weighted design as a design with repeated points. Many known constructions of spherical design first construct designs with repeated points, and then try to apply some separation result to get designs.

Recall that scalars of a levelling space is defined in § 2.2.

Lemma 4.11. *Let $\mathcal{X} = (X, \mu_X)$ be a finite rational-weighted V-design on $\mathcal{Z}$. Then, $\overline{\mathcal{X}} := c\,\mathcal{X}$ is an integer-weighted V-design on $\mathcal{Z}$, where c is a positive integer such that $(c\mu_X)(x)$ is an integer for all $x \in \mathcal{X}$.*

Proof. For arbitrary $f \in V$,

$$\frac{1}{|c\,\mathcal{X}\,|}\int_X f \,\mathrm{d}\, c\mu_X = \frac{1}{|\,\mathcal{X}\,|}\int_X f \,\mathrm{d}\,\mu_X. \qquad \text{(by Eq. (2.2))}$$

Therefore, $\overline{\mathcal{X}}$ is a V-design on $\mathcal{Z}$. Since for every $x \in X$, $(c\mu_X)(x)$ is an integer, $\overline{\mathcal{X}}$ is integer-weighted. □

Remark 4.12. One possible choice for c is

$$c := \mathrm{lcm}_{x \in X}\text{denominator of } \mu_X(x).$$

Corollary 4.13. *Let $\mathcal{X}_s^d$ be a finite semiantipodal rational-weighted rational t-semidesign on $\mathcal{H}_s^d$. Then, $\overline{\mathcal{X}}_s^d$ is a semiantipodal integer-weighted rational t-semidesign on $\mathcal{H}_s^d$.*

Proof. This is an immediate corollary of Lemma 4.11. □

Remark 4.14. By the definition of levelling spaces, an integer-weighted levelling space is automatically finite.

Remark 4.15. In *Step 5*, we apply Corollary 4.13 to explicit designs we constructed in *Step 4*. For those designs, we can write down an explicit formula of a specific multiple of the denominators of $\mu_X(x)$. Therefore, the use of the operations of taking denominators of rational numbers and taking least common multiples of integers can be eliminated.

4.5 *Step 6*: Designs at generic position

The goal of *Step 6* is to put the initial design $\mathcal{Y}^a$ to a "generic position", for our purpose, nonzero first coordinate, so that it would be easier to write down explicit formulas in *Step 7*. Actually, we can relax our requirement and only assume that the first two coordinates are not zeros at the same time. After such relaxation, *Step 6* can be skipped when we choose $a = 1$ in the construction, for instance, in Theorem 1.2(i).

Lemma 4.16. *Let d be a positive integer, and let $x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1} \setminus \{\mathbf{0}\}$ be a point such that $|x_i| \leq 1$. Let δ be a positive real number that is no greater than the absolute values of the nonzero coordinates of x. Let ϵ be an arbitrary rational number such that $0 < \epsilon \leq \delta/4$, and consider the reflection $s_\alpha \in O(d+1, \mathbb{Q})$:*

$$s_\alpha : z \mapsto z - 2\frac{(\alpha, z)}{(\alpha, \alpha)}\alpha, \quad \textit{where} \quad \alpha := (\epsilon^i : i \in [0, d]_{\mathbb{Z}}). \tag{4.7}$$

Then, $s_\alpha(x)$ has a nonzero first coordinate.

Proof. Let $s_\alpha(x)_0$ be the first coordinate of $s_\alpha(x)$, and let $k \in [0, d]_{\mathbb{Z}}$ be the smallest index such that $x_k \neq 0$.

When $x_0 = 0$,

$$\begin{aligned}
|s_\alpha(x)_0| &= \frac{2}{(\alpha, \alpha)} |(\alpha, x)\alpha_0| = \frac{2}{(\alpha, \alpha)} \left| \sum_{i=0}^{d-k} \epsilon^{k+i} x_{k+i} \right| && \text{(by Eq. (4.7))} \\
&\geq \frac{2\epsilon^k}{(\alpha, \alpha)} \left(|x_k| - \sum_{i=1}^{d-k} \epsilon^i |x_{k+i}| \right) \\
&\geq \frac{2\epsilon^k}{(\alpha, \alpha)} \left(\delta - \frac{\epsilon}{1 - \epsilon} \right) && (\text{since } \epsilon < 1,\ |x_k| \geq \delta,\ |x_i| \leq 1) \\
&> 0. && (\text{since } 4\epsilon \leq \delta \leq 1)
\end{aligned}$$

When $x_0 \neq 0$,

$$\begin{aligned}
|s_\alpha(x)_0| &= \frac{1}{(\alpha, \alpha)} |-(\alpha, \alpha)x_0 + 2(\alpha, x)\alpha_0| && \text{(by Eq. (4.7))} \\
&= \frac{1}{(\alpha, \alpha)} \left| \left(2 - \frac{1 - \epsilon^{2d+2}}{1 - \epsilon^2} \right) x_0 - \sum_{i=1}^{d} 2\epsilon^i x_i \right| && \text{(by Eq. (4.7))} \\
&\geq \frac{1}{(\alpha, \alpha)} \left(\frac{1 - 2\epsilon^2 + \epsilon^{2d+2}}{1 - \epsilon^2} |x_0| - \sum_{i=1}^{d} 2\epsilon^i |x_i| \right) \\
&\geq \frac{1}{(\alpha, \alpha)} \left(\frac{1 - 2\epsilon^2}{1 - \epsilon^2}\delta - \frac{2\epsilon}{1 - \epsilon} \right) && (\text{since } \epsilon < 1,\ |x_0| \geq \delta,\ |x_i| \leq 1) \\
&> 0. && (\text{since } 4\epsilon \leq \delta \leq 1)
\end{aligned}$$

Therefore, $s_\alpha(x)$ has a nonzero first coordinate in both cases. ☐

Recall that the image of a levelling space is defined in § 2.2.

Corollary 4.17. *Let $\mathcal{Y}^d$ be an antipodal t-design on $\mathcal{S}^d$ over a field $\mathbb{F} \subseteq \mathbb{R}$. Let $\delta \in \mathbb{F}$ be a positive number that is no greater than the absolute values of nonzero coordinates of points in $\mathcal{Y}^d$. Let ϵ be an arbitrary rational number such that $0 < \epsilon \leq \delta/4$, and consider the reflection $s_\alpha \in O(d+1, \mathbb{Q})$ defined in Eq. (4.7). Then, $s_\alpha \mathcal{Y}^d$ is an antipodal t-design on $\mathcal{S}^d$ over $\mathbb{F}$ such that all points in the design have nonzero first coordinates.*

Proof. Orthogonal transformations over $\mathbb{Q}$ preserve antipodal spherical designs over $\mathbb{F}$, hence $s_\alpha \mathcal{Y}^d$ is an antipodal t-design on $\mathcal{S}^d$ over $\mathbb{F}$. According to Lemma 4.16, for an arbitrary point $x \in \mathcal{Y}^d$, the first coordinate of $s_\alpha(x)$ is nonzero. ☐

Remark 4.18. The finite minimum is used to define δ in Corollary 4.17. However, if we apply our construction to explicit designs, as in Theorem 1.2, the minimum operator can be avoided. For Theorem 1.2(i), (ii), (iii) and (iv), we have $\delta = \operatorname{Im}\zeta_{4(t+1)}$, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$, respectively. An explicit choice of ϵ is $\epsilon := 1/\lceil 4/\delta \rceil$.

4.6 *Step 7*: Twisted products of designs

After *Step 5*, we can view our weighted designs are designs with repeated points. There are various ways to separate the repeated points. Using topology to separate, we prove the existence of designs; using constructive topology to separate, we get computable designs; [8] uses analytic number theory to separate, which gives an algorithm to find designs; *Step 7* uses extra dimensions to separate, which lead us to explicit designs.

Recall that in § 2.2 we define integer-weighted and $\underline{1}$-weighted levelling spaces, and we define the maps, sums, products and images of levelling spaces. It is easy to see that, for an integer-weighted levelling space $\mathcal{X} = (X, \mu_X)$, we have

$$\mathcal{X} = \sum_{x\in X}\sum_{i=1}^{\mu_X(x)} \boldsymbol{x}, \quad X = \bigcup_{x\in X}\bigcup_{i=1}^{\mu_X(x)} \{x\} \quad \text{and} \quad \mu_X = \sum_{x\in X}\sum_{i=1}^{\mu_X(x)} \underline{1}_x, \tag{4.8}$$

where $\boldsymbol{x}$ is understood as the levelling space with support $\{x\}$ equipped with the counting measure $\underline{1}_x$ on $\{x\}$.

For a levelling space $\mathcal{Z}$, let $\operatorname{Aut}(\mathcal{Z})$ be the set of all automorphisms of $\mathcal{Z}$.

Definition 4.19. Let $\mathcal{Y} \subseteq \mathcal{Z}$ be a levelling space and $\mathcal{X} = (X, \mu_X)$ an integer-weighted levelling space. Let $\xi : \mathbb{N} \to \operatorname{Aut}(\mathcal{Z})$ be a map. The *twisted product* of $\mathcal{Y}$ and $\mathcal{X}$ with respect to ξ is the levelling space

$$\mathcal{Y} \rtimes_\xi \mathcal{X} := \sum_{x\in X}\sum_{i=1}^{\mu_X(x)} \xi(i)\,\mathcal{Y} \times \boldsymbol{x}.$$

In other words, the twisted product is $\mathcal{Y} \rtimes_\xi \mathcal{X} := (Y \rtimes_\xi X, \mu_Y \rtimes_\xi \mu_X)$, where

$$Y \rtimes_\xi X := \bigcup_{x\in X}\bigcup_{i=1}^{\mu_X(x)} \xi(i)Y \times \{x\} \quad \text{and} \quad \mu_Y \rtimes_\xi \mu_X := \sum_{x\in X}\sum_{i=1}^{\mu_X(x)} \xi(i)_*\mu_Y \times \underline{1}_x.$$

Since $\mathcal{X}$ is integer-weighted and the total measure is finite, all sums are finite sums.

The total measures satisfy the equation $|\,\mathcal{Y} \rtimes_\xi \mathcal{X}\,| = |\,\mathcal{Y}\,| \cdot |\,\mathcal{X}\,|$. When ξ is the constant function with image the identity automorphism of $\mathcal{Z}$, the twisted product is just the ordinary product, namely, $\mathcal{Y} \rtimes_\xi \mathcal{X} = \mathcal{Y} \times \mathcal{X}$.

Theorem 4.20. *Let $\mathcal{Z}_0$ and $\mathcal{Z}_1$ be levelling spaces, and let V_0 and V_1 be real vector spaces of continuous integrable functions on $\mathcal{Z}_0$ and $\mathcal{Z}_1$, respectively. Let $\mathcal{X}_0$ be a V_0-design on $\mathcal{Z}_0$ and $\mathcal{X}_1$ an integer-weighted V_1-design on $\mathcal{Z}_1$. Let $\xi : \mathbb{N} \to \operatorname{Aut}(\mathcal{Z})$ be a map and assume that $\operatorname{Aut}(\mathcal{Z})$ preserves V_0. Then, the twisted product $\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1$ is a $V_0 \otimes V_1$-design on $\mathcal{Z}_0 \times \mathcal{Z}_1$. Moreover, if $\mathcal{X}_0$ is $\underline{1}$-weighted and $\bigcup_{i\in\mathbb{N}} \xi(i)X_0$ is a disjoint union, then $\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1$ is $\underline{1}$-weighted.*

Proof. For arbitrary $f_0 \in V_0$ and $f_1 \in V_1$,

$$\frac{1}{|\,\mathcal{X}_0 \rtimes_\xi \mathcal{X}_1\,|}\int_{X_0 \rtimes_\xi X_1} f_0 \otimes f_1 \,\mathrm{d}\,\mu_{X_0} \rtimes_\xi \mu_{X_1}$$

$$
\begin{aligned}
=&\frac{1}{|\mathcal{X}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\int_{\xi(i)X_0\times x_1} f_0\otimes f_1\,\mathrm{d}\,\xi(i)_*\mu_{X_0}\times \underline{1}_{x_1}\\
&\hfill\text{(by Definition 4.19 and Eq. (2.3))}\\
=&\frac{1}{|\mathcal{X}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{\xi(i)X_0} f_0\,\mathrm{d}\,\xi(i)_*\mu_{X_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) \qquad \text{(by Eq. (2.4))}\\
=&\frac{1}{|\mathcal{X}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{X_0} \xi(i)^* f_0\,\mathrm{d}\,\mu_{X_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) \qquad \text{(by Eq. (2.5))}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{Z_0} \xi(i)^* f_0\,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right)\\
&\hfill\text{(since } \xi(i)^* f_0\in V_0,\ \mathcal{X}_0 \text{ is a } V_0\text{-design)}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{\xi(i)Z_0} f_0\,\mathrm{d}\,\xi(i)_*\mu_{Z_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right) \qquad \text{(by Eq. (2.5))}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\left(\int_{Z_0} f_0\,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{x_1} f_1\,\mathrm{d}\,\underline{1}_{x_1}\right)\\
&\hfill\text{(since } \xi(i) \text{ is an automorphism)}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{X}_1|}\left(\int_{Z_0} f_0\,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{X_1} f_1\,\mathrm{d}\,\mu_{X_1}\right) \qquad \text{(by Eqs. (2.3) and (4.8))}\\
=&\frac{1}{|\mathcal{Z}_0||\mathcal{Z}_1|}\left(\int_{Z_0} f_0\,\mathrm{d}\,\mu_{Z_0}\right)\left(\int_{Z_1} f_1\,\mathrm{d}\,\mu_{Z_1}\right) \qquad \text{(since } f_1\in V_1,\ \mathcal{X}_1 \text{ is a } V_1\text{-design)}\\
=&\frac{1}{|\mathcal{Z}_0\times\mathcal{Z}_1|}\int_{Z_0\times Z_1} f_0\otimes f_1\,\mathrm{d}\,\mu_{Z_0}\times\mu_{Z_1}. \qquad \text{(by Eq. (2.4))}
\end{aligned}
$$

Therefore, the twisted product is a $V_0\otimes V_1$-design on $\mathcal{Z}_0\times\mathcal{Z}_1$.

When $\mathcal{X}_0$ is $\underline{1}$-weighted, the levelling space $\xi(i)\,\mathcal{X}_0\times\boldsymbol{x}_1$ is also $\underline{1}$-weighted. Since $\bigcup_{i\in\mathbb{N}}\xi(i)X_0$ is a disjoint union, $\mathcal{X}_0\rtimes_\xi\mathcal{X}_1 = \sum_{x_1\in X_1}\sum_{i=1}^{\mu_{X_1}(x_1)}\xi(i)\,\mathcal{X}_0\times\boldsymbol{x}_1$ is a disjoint union as well. Therefore, $\mathcal{X}_0\rtimes_\xi\mathcal{X}_1$ is $\underline{1}$-weighted. □

Consider the metric space S^1. Let $\mathrm{SO}(2,\mathbb{R})_{<s}$ be the local group of all rotations g such that $\mathrm{dist}(x,gx)<s$ for $x\in S^1$. Recall from Definition 3.4(i) that, for a subset $X\subseteq S^1$, $\mathrm{dist}_{\min}X$ is the infimum of distances between any two distinct points of X.

Lemma 4.21. *Let X be a subset of S^1 and $s:=(\mathrm{dist}_{\min}X)/2$. Then, $\bigcup_g gX$ is a disjoint union where g runs over $\mathrm{SO}(2,\mathbb{R})_{<s}$.*

Proof. Let $g_0,g_1\in\mathrm{SO}(2,\mathbb{R})_{<s}$ be two arbitrary distinct rotations. For distinct $x_0,x_1\in X$,

$$
\begin{aligned}
\mathrm{dist}(g_0x_0,g_1x_1)&\geq\mathrm{dist}(g_0x_0,g_0x_1)-\mathrm{dist}(x_1,g_0x_1)-\mathrm{dist}(x_1,g_1x_1)\\
&>\mathrm{dist}(x_0,x_1)-\mathrm{dist}_{\min}(X)\geq 0,
\end{aligned}
$$

hence $g_0x_0\neq g_1x_1$. For $x_0=x_1\in X$, since $g_0\neq g_1$, $g_0x_0\neq g_1x_1$. Therefore, g_0X and g_1X are disjoint. □

Lemma 4.22. *Let $X \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$ be a finite subset. Let $p : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathcal{S}^1$ be the projection $(x_0, x_1) \mapsto \left(\frac{x_0}{\sqrt{x_0^2+x_1^2}}, \frac{x_1}{\sqrt{x_0^2+x_1^2}}\right)$. Consider the map*

$$\xi_X^1 : \mathbb{N} \to \mathrm{O}(2,\mathbb{Q}) \subseteq \mathrm{Aut}(\mathcal{S}^1),$$
$$i - 1 \mapsto \frac{1}{i^2+s^2} \begin{bmatrix} i^2 - s^2 & -2is \\ 2is & i^2 - s^2 \end{bmatrix},$$

where $s := \max\{\frac{1}{n+1} : \frac{1}{n+1} < (\mathrm{dist}_{\min} p(X))/2, n \in \mathbb{N}\} \in \mathbb{Q}$. Then, $\bigcup_{i\in\mathbb{N}} \xi_X^1(i)X$ is a disjoint union.

Proof. It is easy to check that the image of ξ_X^1 is in $\mathrm{SO}(2,\mathbb{R})_{<s}$, hence it is also in $\mathrm{SO}(2,\mathbb{Q})_{<\mathrm{dist}_{\min} p(X))/2}$. Applying Lemma 4.21 to $p(X) \subseteq \mathcal{S}^1$, we know that $\bigcup_{i\in\mathbb{N}} \xi_X^1(i)p(X)$ is a disjoint union, hence $\bigcup_{i\in\mathbb{N}} \xi_X^1(i)X$ is a disjoint union as well. □

Remark 4.23. The use of the minimum distance $\mathrm{dist}_{\min}$ and maximum $\max\{\dots\}$ in Lemma 4.22 can be avoided. For explicit X, for instance the X's obtained from well-known spherical designs in Theorem 1.2, the minimum distance $\mathrm{dist}_{\min} X$ is known. Instead of maximum, we can use ceiling/floor to define s.

Corollary 4.24. *Let d be a positive integer. Let $X \subseteq \mathcal{S}^d$ be a finite subset such that every point in X has a nonzero first coordinate. Let $p : \mathcal{S}^d \to \mathbb{R}^2$ be the projection $(x_0, \dots, x_d) \mapsto (x_0, x_1)$. Consider the map*

$$\xi_X^d := j \circ \xi_{p(X)}^1 : \mathbb{N} \to O(d+1,\mathbb{Q}) \subseteq \mathrm{Aut}(\mathcal{S}^d),$$

where $j : \mathrm{O}(2,\mathbb{Q}) \to \mathrm{O}(d+1,\mathbb{Q})$ is the inclusion corresponding to p. Then, $\bigcup_{i\in\mathbb{N}} \xi_X^d(i)X$ is a disjoint union.

Proof. Since every point in X has a nonzero first coordinate, $p(X) \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$. By Lemma 4.22, $\bigcup_{i\in\mathbb{N}} \xi_{p(X)}^1(i)p(X) = \bigcup_{i\in\mathbb{N}} p(\xi_X^d(i)X)$ is a disjoint union, hence $\bigcup_{i\in\mathbb{N}} \xi_X^d(i)X$ is a disjoint union. □

Corollary 4.25. *Let a be a positive integer and let b be a natural number. Let $\mathcal{Y}^a$ be an antipodal t-design on $\mathcal{S}^a$ over $\mathbb{F}$ such that every point in $\mathcal{Y}^a$ has a nonzero first coordinate, and let $\overline{\mathcal{X}}_a^b$ a semiantipodal integer-weighted rational t-semidesign on $\mathcal{H}_a^b$. Let $\xi := \xi_{\mathcal{Y}^a}^a$, which is defined in Corollary 4.24. Then, the twisted product $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^b$, regarded as a subspace of $\mathcal{S}^{a+b}$, is an antipodal t-design on $\mathcal{S}^{a+b}$ over $\mathbb{F}$.*

Proof. We use the similar ideas to Lemma 4.9. We label the coordinates of S^a, H^b and S^{a+b} as follows

$$S^a = \{(x_0, \dots, x_a)\}, \quad H^b = \{(y_0, \dots, y_b)\}, \quad S^{a+b} = \{(z_0, \dots, z_{a+b})\}.$$

Since $\mathcal{Y}^a$ is antipodal and $\overline{\mathcal{X}}_a^b$ is semiantipodal, using the fact that orthogonal transformations preserve antipodal map, the twisted product $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}_a^b$, regarded as a subspace of $\mathcal{S}^{a+b}$ by Remark 4.4, is antipodal and $[0,a]$-antipodal (see Definition 2.4 for definition of antipodal).

Recall that $\mathcal{P}^{t,a}$ is defined in Eq. (2.7), and $\mathcal{P}^{t,a}$ is a subfunctor of $\mathcal{P}^t$. Consider the dominant open embedding $\iota_{a,b}$ in Proposition 4.2(i). The comorphism $\iota_{a,b}^*$ gives an inclusion

$$\begin{aligned} \iota_{a,b}^* : \mathcal{P}^{t,a}[S^{a+b}] &\hookrightarrow \mathcal{P}^t[S^a] \otimes \mathcal{P}^{t,0}[H^b], \\ z_i &\mapsto \begin{cases} x_i \otimes y_0, & \text{if } i \in [1,a]_{\mathbb{Z}}, \\ 1 \otimes y_{i-a}, & \text{if } i \in [a+1, a+b]_{\mathbb{Z}}. \end{cases} \end{aligned}$$

Similar to Lemma 4.9, although y_0 is not in $\mathcal{P}^{t,0}[H^b]$, any monomial in $\mathcal{P}^{t,a}[S^{a+b}]$ maps to some monomial with even degree in y_0, hence in $\mathcal{P}^{t,0}[H^b]$.

Since $\mathcal{Y}^a$ is a $\underline{1}$-weighted $\mathcal{P}^t[S^a]$-design on $\mathcal{S}^a$ and $\overline{\mathcal{X}}^b_a$ is an integer-weighted $\mathcal{P}^{t,0}[H^b]$-design on $\mathcal{H}^b_a$, by Theorem 4.20, $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}^b_a$ is a $\underline{1}$-weighted $\mathcal{P}^t[S^a] \otimes \mathcal{P}^{t,0}[H^b]$-design on $\mathcal{S}^a \times \mathcal{H}^b_a$. According to the inclusion $\iota^*_{a,b}$, the twisted product $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}^b_a$, regarded a subspace of $\mathcal{S}^{a+b}$, is a $\underline{1}$-weighted $\mathcal{P}^{t,a}[S^{a+b}]$-design on $\mathcal{S}^{a+b}$. Therefore, by Lemma 2.6, $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}^b_a$ is a $\underline{1}$-weighted t-design on $\mathcal{S}^{a+b}$, hence a t-design on $\mathcal{S}^{a+b}$.

Since $\overline{\mathcal{X}}^b_a$ is rational and ξ has rational coefficients, the twisted product preserves the field. Thus, $\mathcal{Y}^a \rtimes_\xi \overline{\mathcal{X}}^b_a$ is a design over $\mathbb{F}$. □

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