RATIONAL DESIGNS

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ABSTRACT. The existence of designs on a path-connected space was proved by Seymour and Zaslavsky. In this paper, under certain necessary conditions, we establish the existence of designs consisting of rational points on an algebraically path-connected space. Consequently, we show that there exist rational designs on rational convex polytopes and spherical designs consisting of points whose coordinates are rational numbers except possibly for the first coordinate.

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1. INTRODUCTION

Designs are subsets of a given space that approximate the whole space nicely. There are primarily two classes of designs, combinatorial designs and geometric designs. The former class includes classical combinatorial t-designs [RCW75, Kee14], and their generalizations, for instance relative designs on association schemes [Del73, BB12, BBST15]. The latter class is also known as cubature formula [Möl79, GS81] and averaging sets [SZ84]. Geometric designs were first introduced by Delsarte, Goethals and Seidel in [DGS77]. Typical examples include spherical designs and their variations, such as interval designs, Euclidean designs [NS88, DS89], designs on Grassmannians [BCN02, BBC04], complex spherical designs [RS14] and designs on polynomial spaces [God88, BBD99]. In this paper we will focus on geometric designs. For simplicity, in the introduction we only give below a definition for designs on subsets of Euclidean spaces, which is also adopted in [SZ84]. We will define and discuss designs in a more general setup in § 2.2.

Definition 1.1. Let Z be a nonempty topological subspace of $\mathbb{R}^d$ and μ_Z a measure on Z such that every nonempty open set has positive measure and the total measure $\mu_Z(Z)$ is finite. Let t be a natural number. A *t-design on $\mathcal{Z} := (Z, \mu_Z)$* is a nonempty finite set $X \subseteq Z$ such that

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{\mu_Z(Z)}\int_Z f \,\mathrm{d}\,\mu_Z, \tag{1.1}$$

for every polynomial $f \in \mathbb{R}[x_1, \dots, x_d]$ of degree at most t. When $X \subseteq \mathbb{F}^d$ for some subfield $\mathbb{F}$ of $\mathbb{R}$, we call the t-design X a *t-design over* $\mathbb{F}$. A *rational t-design* is a t-design over $\mathbb{Q}$.

There is a large body of literature on designs over $\mathbb{R}$. Seymour and Zaslavsky [SZ84] showed that for all natural numbers t, t-designs on $\mathcal{Z}$ always exist as long as Z is path-connected. In particular, *spherical t-designs*, that is, t-designs on the d-dimensional real unit sphere $\mathcal{S}^d$ endowed with the spherical measure, always exist for all natural numbers t and d. Spherical designs can be constructed from interval designs [RB91, Wag91]. The spherical designs whose sizes achieve certain lower bounds [DGS77] are classified except for $t \in \{4, 5, 7\}$ [BD79, BD80]. Small spherical designs on $\mathcal{S}^2$ can be numerically constructed [HS96, CFL11]. All possible asymptotic sizes of spherical designs are found [BRV13]. We refer to the survey by Eiichi Bannai and Etsuko Bannai [BB09] for further information.

Kuperberg [Kup05] constructed interval designs over $\mathbb{Q}^{\mathrm{alg}} \cap \mathbb{R}$, where $\mathbb{Q}^{\mathrm{alg}}$ is the algebraic closure of $\mathbb{Q}$. On the unit circle $\mathcal{S}^1$, a regular $(t+1)$-gon gives a spherical t-design over the field $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$, where $\mathbb{Q}^{\mathrm{ab}}$ is the abelian closure of $\mathbb{Q}$. In a forthcoming paper, we show that we can lift a design on $\mathcal{S}^1$ over $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$ to a design on $\mathcal{S}^d$ over the same field using a method that generalizes the one in [RB91, Wag91].

The following two famous examples of rational spherical designs were discovered by Venkov [Ven84].

Example 1.2. For an integral lattice Λ and a natural number m, let $\Lambda_m := \{x \in \Lambda : \|x\|^2 = m\}$ be the shell of lattice points of norm m.

(i) Let $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^8$ be the E_8-lattice. For every $m \in \mathbb{Z}_{\geq 1}$, $\frac{1}{2m}\Lambda_{4m^2}$ is a rational spherical 7-design on $\mathcal{S}^7$.

(ii) Let $\Lambda \subseteq \frac{1}{\sqrt{8}}\mathbb{Z}^{24}$ be the Leech lattice. For every $m \in \mathbb{Z}_{\geq 2}$, $\frac{1}{\sqrt{2}m}\Lambda_{2m^2}$ is a rational spherical 11-design on $\mathcal{S}^{23}$.

Conditions 1.3 and 1.4 below give two natural necessary conditions on $\mathcal{Z} = (Z, \mu_Z)$ for the existence of rational designs. The necessity is proved in Propositions 3.3 and 3.5.

Condition 1.3. The rational points $Z \cap \mathbb{Q}^d$ in Z are dense in Z.

Condition 1.4. For every polynomial $f \in \mathbb{Q}[x_1, \dots, x_d]$, $\frac{1}{\mu_Z(Z)} \int_Z f \, \mathrm{d}\,\mu_Z$ is a rational number.

Eiichi Bannai was the first to ask the question about the existence of rational t-designs on a given $\mathcal{Z}$ for all $t \in \mathbb{N}$. The main result of the paper, Theorem 1.6 below, states that, assuming Conditions 1.3 and 1.4, if the space $\mathcal{Z}$ has good algebraic path-connectivity defined in Definition 1.5, then there exist rational designs on $\mathcal{Z}$.

Definition 1.5. Let $Z \subseteq \mathbb{R}^d$ be a subset and $\mathbb{F} \subseteq \mathbb{R}$ a subfield. The set Z is called $\mathbb{F}$-*algebraically path-connected* provided that for every finite subset $X \subseteq Z \cap \mathbb{F}^d$, there exists a polynomial map $p \in \mathbb{F}[x]^d$ such that $X \subseteq p((0,1) \cap \mathbb{F}) \subseteq Z \cap \mathbb{F}^d$.

Theorem 1.6. *Let $\mathcal{Z}$ be as in Definition 1.1, and let t be a natural number. Assume that $\mathcal{Z}$ satisfies Conditions 1.3 and 1.4. If $\mathcal{Z}$ is $\mathbb{Q}$-algebraically path-connected, then there exists a natural number n_0 such that for every natural number $n \geq n_0$, there exist rational t-designs on $\mathcal{Z}$ of size n.*

Theorem 1.6 will follow from the stronger Theorem 7.1, where we provide in addition an asymptotic lower bound for the number of rational designs. Both Theorems 1.6 and 7.1 are proved in § 7.1.

Using approximation theory, we give a simple sufficient condition for $\mathbb{F}$-algebraic path-connectivity in Theorem 1.7 below, which is proved in § 6.1.

Theorem 1.7. *Let $Z \subseteq \mathbb{R}^d$ be an open connected subset. For every subfield $\mathbb{F} \subseteq \mathbb{R}$, Z is $\mathbb{F}$-algebraically path-connected.*

Using Theorem 1.6, we show the existence of rational designs on rational convex polytopes in Theorem 1.8, which is proved in § 7.2.

Theorem 1.8. *Let $Z \subseteq \mathbb{R}^d$ be a d-dimensional convex polytope whose vertices are in $\mathbb{Q}^d$, and μ_Z the Lebesgue measure. Let t be a natural number. Then, there exists a natural number n_0 such that for every natural number $n \geq n_0$, there exist rational t-designs on (Z, μ_Z) of size n. In particular, there exist rational t-designs on the unit interval $[0, 1]$.*

The existence of rational spherical designs is a longstanding open problem. Proposition 6.3 shows that $\mathcal{S}^d$ is not $\mathbb{F}$-algebraically path-connected for any subfield $\mathbb{F} \subseteq \mathbb{R}$, so our main result Theorem 1.6 cannot be applied directly. In the case of $\mathcal{S}^d$, we can only show the existence of "almost rational" spherical designs, as in our next theorem.

Theorem 1.9. *Let t and d be natural numbers. There exists a natural number n_0 such that for every even natural number $n \geq n_0$, there exist spherical t-designs on $\mathcal{S}^d$ of size n where for each point in the design, all its coordinates are rational numbers except possibly for the first coordinate. In particular, there exist spherical t-designs of size n over the field $\mathbb{Q}(\{\sqrt{q} : q \text{ prime}\})$.*

The proof of Theorem 1.9, presented in § 7.2, relies on Theorem 1.6. Note that Theorem 1.9 also provides a new proof of the existence of spherical designs.

The paper is organized as follows. In § 2, we introduce the concept of *levelling space*, which provides a framework to talk about designs in a general setup. We then discuss properties of designs and construct weighted designs using convex geometry. In § 3, we prove the necessity of Conditions 1.3 and 1.4, and also show that they are sufficient to construct certain weighted rational designs. In § 4, we analyze the possible total measures of integer-weighted designs. In § 5, the analytic number theoretic arguments which we need in § 7 are given. § 6 is concerned with algebraic path-connectivity and contains a proof of Theorem 1.7 via approximation theory. At the end, in § 7, we prove Theorem 7.1, whence the main result Theorem 1.6 follows, together with its corollaries.

Notation. For a subset $S \subseteq \mathbb{R}$ and a real interval $I \subseteq \mathbb{R}$, let $I_S := I \cap S$. For instance, for $a, b \in \mathbb{R}$, $(a, b)_{\mathbb{Q}}$ consists of all rational numbers in (a, b). Another example is $[0, t]_{\mathbb{Z}}$, which consists of all natural numbers no greater than t. The d-dimensional real unit sphere equipped with the spherical measure is denoted by $\mathcal{S}^d$ and the unit open interval $(0, 1)$ equipped with the Lebesgue measure is denoted by $\mathcal{I}$.

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2. Levelling spaces and designs

In this section, we discuss certain topological measure spaces which we call *levelling spaces*. Some basic properties and constructions of levelling spaces are given in § 2.1. In § 2.2, we first define an equivalence relation of levelling spaces, and then use it to define designs on a levelling space. In § 2.3, we use some results in convex geometry to construct weighted designs on arbitrary levelling spaces.

2.1. Levelling spaces and their properties.

A *strictly positive measure space* $\mathcal{X} = (X, \mu_X)$ is a Hausdorff topological space X equipped with a measure μ_X such that every nonempty open set has positive measure. In particular for a discrete space, it means that every point has positive measure. The set X is said to be the *support* of $\mathcal{X}$.

Definition 2.1. A *levelling space* is a nonempty strictly positive measure space of finite total measure. The *size* of a levelling space is the cardinality of its support. A levelling space is called *finite* if its size is finite.

Throughout the whole paper, a levelling space $\mathcal{X}$ is always written in calligraphic font $\mathcal{X}$, in order to be distinguished from its support, the topological space X. For two levelling spaces $\mathcal{X}$ and $\mathcal{Z}$, by saying $\mathcal{X}$ *is a subspace of* $\mathcal{Z}$, denoted by $\mathcal{X} \subseteq \mathcal{Z}$, we only mean that the support X of $\mathcal{X}$ is a topological subspace of the support Z of $\mathcal{Z}$ without assuming any relation between the measures μ_X and μ_Z. Similarly if $\mathcal{X}$ is a levelling space and Z a topological space, we write $\mathcal{X} \subseteq Z$ to mean that X is a topological subspace of Z.

Definition 2.2. Let $k \subseteq \mathbb{R}$ be a subset. A levelling space $\mathcal{X} = (X, \mu_X)$ is *k-weighted* if the image of μ_X is in k.

Usually, we take k to be either $\mathbb{Z}$ or $\mathbb{Q}$, and call the levelling space to be *integer-weighted* or *rational-weighted*, respectively. Recall that, as a measure space, the image of μ_X is always nonnegative. A finite levelling space $\mathcal{X}$ is just a nonempty weighted set (X, ω_X) where $\omega_X : X \to \mathbb{R}_{>0}$. In particular, a finite integer-valued levelling space can be viewed as a nonempty finite multi-set.

Now, we introduce some operations on levelling spaces.

Definition 2.3. Let $\mathcal{X} = (X, \mu_X)$ and $\mathcal{Y} = (Y, \mu_Y)$ be two levelling spaces whose subtopologies on the set intersection $X \cap Y$ agree. The *sum of $\mathcal{X}$ and $\mathcal{Y}$*, denoted by $\mathcal{X} + \mathcal{Y}$, is defined to be $(X \cup Y, \mu_X + \mu_Y)$, where $X \cup Y$ is the topological union of X and Y, and $\mu_X + \mu_Y$ is the sum of the measures μ_X and μ_Y. More precisely, a subset $E \subseteq X \cup Y$ is measurable in $\mathcal{X} + \mathcal{Y}$ if and only if $E \cap X$ and $E \cap Y$ are measurable in $\mathcal{X}$ and $\mathcal{Y}$ respectively, and $(\mu_X + \mu_Y)(E) := \mu_X(E \cap X) + \mu_Y(E \cap Y)$ for such E.

Definition 2.4. Let $\mathcal{X} = (X, \mu_X)$ be a levelling space and let c be a positive real number. We set $c\,\mathcal{X} := (X, c\,\mu_X)$, where $c\,\mu_X$ is a constant scalar of μ_X.

Definition 2.5. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. For a levelling space $\mathcal{W} = (W, \mu_W)$ with $W \subseteq X$, the *image of $\mathcal{W}$ under f* is defined to be $f(\mathcal{W}) := (f(W), f_*\mu_W)$, where $f(W)$ is the image of W under f as a topological space, and $f_*\mu_W$ is the pushfoward of the measure μ_W. That is to say, a subset $E \subseteq f(W)$ is measurable in $f(\mathcal{W})$ if and only if $f^{-1}(E) \cap W$ is measurable in $\mathcal{W}$ and $(f_*\mu_W)(E) := \mu_W(f^{-1}(E) \cap W)$ for such E.

Lemma 2.6 shows that the sums, positive scalars, and images of levelling spaces are still levelling spaces, and it also provides some other constructions to be used later.

Lemma 2.6. *Let X and Y be two topological spaces. The following statements hold.*

- (i) *Suppose that X and Y agree on $X \cap Y$. For levelling spaces $\mathcal{X}$ with support X and $\mathcal{Y}$ with support Y, their sum $\mathcal{X} + \mathcal{Y}$ is a levelling space.*
- (ii) *Let $\mathcal{X}$ be a levelling space, and let c be a positive real number. Then, $c\,\mathcal{X}$ is a levelling space.*
- (iii) *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be levelling spaces whose supports agree on pairwise intersections, and let $c_1, \ldots, c_n$ be nonnegative real numbers that are not all zeros. Then, the linear combination $\sum_{i=1}^{n} c_i\,\mathcal{X}_i$ is a levelling space.*
- (iv) *Let $f : X \to Y$ be a continuous map. For a levelling space $\mathcal{W} \subseteq X$, the image $f(\mathcal{W})$ is a levelling space.*
- (v) *Let $f : X \to Y$ be a continuous map such that each of its fibers is finite. For a finite levelling space $\mathcal{W}$ with support $W \subseteq f(X)$, there exists a levelling space $f^{-1}(\mathcal{W})$ with support $f^{-1}(W)$ such that $f(f^{-1}(\mathcal{W})) = \mathcal{W}$ and all points in each fiber have the same measure.*
- (vi) *Let $f : X \to Y$ be an n-fold covering. For a finite levelling space $\mathcal{W} \subseteq Y$ equipped with the counting measure, the levelling space $nf^{-1}(\mathcal{W})$ is equipped with the counting measure.*

Proof. (i): Let E be an open set in $X \cup Y$. The restriction $E \cap X$ and $E \cap Y$ are open in X and Y, respectively. Then, we have $(\mu_X + \mu_Y)(E) = \mu_X(E \cap X) + \mu_Y(E \cap Y) > 0$. Moreover, the total measure of $\mathcal{X} + \mathcal{Y}$ is the sum of total measures of X and Y, hence finite. Therefore, $\mathcal{X} + \mathcal{Y}$ is a levelling space.

(ii): A positive scalar of a strictly positive measure with finite total measure is still a strictly positive measure with finite total measure, and the result follows.

(iii): Corollary of (i) and (ii).

(iv): Let E be an open set in $f(W)$. Since f is continuous, the set $f^{-1}(E) \cap W$ is open in W. Then, $(f_*\mu_W)(E) = \mu_W(f^{-1}(E) \cap W) > 0$. The total measure of $f(\mathcal{W})$ is the same with the total measure of $\mathcal{W}$, hence finite. Therefore, $f(\mathcal{W})$ is a levelling space.

(v) Let $f^{-1}(\mathcal{W})$ be the levelling space with support $f^{-1}(W)$ equipped with the measure μ such that $\mu(x) := \frac{\mu_W(f(x))}{|f^{-1}(f(x))|}$ for every point $x \in f^{-1}(W)$, which is well-defined since both W and $f^{-1}(W)$ are finite. It is then clear that $f(f^{-1}(\mathcal{W})) = \mathcal{W}$.
(vi): The result follows from the construction of $f^{-1}(\mathcal{W})$ in (v). □

2.2. **Equivalent levelling spaces and designs.** We say that a function f is *integrable* on a levelling space $\mathcal{X} = (X, \mu_X)$ if $f|_X$ is μ_X-integrable on X. In this section, we fix a levelling space $\mathcal{Z}$ and a finite dimensional real vector space V of continuous integrable real-valued functions on $\mathcal{Z}$.

Definition 2.7. Let $\mathcal{X} \subseteq \mathcal{Z}$ be a levelling space such that all functions in V are integrable on $\mathcal{X}$. For each $f \in V$, the *centroid of f on $\mathcal{X}$* is

$$\operatorname{centroid}_{\mathcal{X}} f := \frac{1}{\mu_X(X)} \int_X f \, \mathrm{d}\, \mu_X.$$

The *centroid of V on $\mathcal{X}$*, denoted by $\operatorname{centroid}_{\mathcal{X}} V$, is defined to be an element in the *dual space* V^*:

$$\begin{array}{rccc} \operatorname{centroid}_{\mathcal{X}} V: & V & \to & \mathbb{R} \\ & f & \mapsto & \operatorname{centroid}_{\mathcal{X}} f. \end{array}$$

Definition 2.8. Two levelling spaces $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{Z}$ are *V-equivalent*, provided that $\operatorname{centroid}_{\mathcal{X}} V = \operatorname{centroid}_{\mathcal{Y}} V$.

Clearly, V-equivalence is an equivalence relation of levelling spaces that are contained in $\mathcal{Z}$.

Lemma 2.9. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n \subseteq \mathcal{Z}$ be levelling spaces that are V-equivalent to a levelling space $\mathcal{X} \subseteq \mathcal{Z}$, and let $c_1, \ldots, c_n$ be nonnegative real numbers that are not all zeros. Then, the linear combination $\sum_{i=1}^n c_i \mathcal{X}_i$ is also V-equivalent to $\mathcal{X}$.*

Proof. By Lemma 2.6(iii), the linear combination $\mathcal{Y} := \sum_{i=1}^n c_i \mathcal{X}_i$ is a levelling space. For every $f \in V$,

$$\operatorname{centroid}_{\mathcal{Y}} f = \frac{1}{\mu_Y(Y)} \int_Y f \, \mathrm{d}\, \mu_Y = \frac{\sum_{i=1}^n c_i \int_{X_i} f \, \mathrm{d}\, \mu_{X_i}}{\sum_{i=1}^n c_i \mu_{X_i}(X_i)}.$$

For each i, $\frac{\int_{X_i} f \, \mathrm{d}\, \mu_{X_i}}{\mu_{X_i}(X_i)} = \operatorname{centroid}_{\mathcal{X}_i} f = \operatorname{centroid}_{\mathcal{X}} f$ by V-equivalence between $\mathcal{X}_i$ and $\mathcal{X}$. Therefore, $\operatorname{centroid}_{\mathcal{Y}} f = \operatorname{centroid}_{\mathcal{X}} f$, hence the result. □

Lemma 2.10. *Let $p\colon \mathcal{Z}' \to \mathcal{Z}$ be a continuous map between levelling spaces $\mathcal{Z}'$ and $\mathcal{Z}$, and let $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{Z}'$ be levelling spaces. Then, $p(\mathcal{X}_1)$ and $p(\mathcal{X}_2)$ are V-equivalent levelling spaces in $\mathcal{Z}$ if and only if $\mathcal{X}_1$ and $\mathcal{X}_2$ are p^*V-equivalent levelling spaces in $\mathcal{Z}'$, where $p^*V := \{f \circ p : \ f \in V\}$.*

Proof. It is easy to check that $\operatorname{centroid}_{p(\mathcal{X})} f = \operatorname{centroid}_{\mathcal{X}} f \circ p$ for every $f \in V$ and $\mathcal{X} \in \{\mathcal{X}_1, \mathcal{X}_2\}$, hence the result. □

Definition 2.11. A levelling space $\mathcal{X} \subseteq \mathcal{Z}$ is a *weighted V-design on $\mathcal{Z}$* if $\mathcal{X}$ is V-equivalent to $\mathcal{Z}$. An *V-design on $\mathcal{Z}$* is a weighted V-design $\mathcal{X}$ on $\mathcal{Z}$ where the associated measure μ_X is the counting measure, namely $\mu_X(x) = 1$ for every $x \in \mathcal{X}$.

The levelling space $\mathcal{Z}$ itself is clearly a weighted V-design on $\mathcal{Z}$.

2.3. **Existence of weighted designs.** Recall that we have fixed a levelling space $\mathcal{Z}$ and a finite dimensional real vector space V of continuous integrable real-valued functions on $\mathcal{Z}$. We equip V with the finest locally convex topology on V, and view V as a real topological vector space. The dual space V^* is then also a real topological vector space with $\dim V^* = \dim V$. If we choose a basis for V and use it and its dual basis to identify V and V^* with $\mathbb{R}^{\dim V}$, then the finest locally convex topologies on V and V^* are homeomorphic to the natural topology on $\mathbb{R}^{\dim V}$. The topological structure of V^* allows us to talk about limit, interior, and dimension of some subsets of V^*.

Definition 2.12. A subset $X \subseteq \mathcal{Z}$ is called *V-nondegenerate* provided that the restriction $V|_X := \{f|_X : f \in V\}$ satisfies $\dim V|_X = \dim V$.

In other words, X is V-nondegenerate if and only if the natural epimorphism $V \twoheadrightarrow V|_X$ is an isomorphism.

Let S be a subset of the real topological vector space V^*. The *convex cone of S*, *convex hull of S* and the *affine space generated by S* are the sets of all nonnegative, convex and affine linear combinations of S and are denoted by $\operatorname{cone} S$, $\operatorname{conv} S$ and $\operatorname{aff} S$, respectively. Let $\operatorname{int} S$ be the *interior of S in V^**, and $\operatorname{reint} S$ be the *relative interior of S*, that is, the interior of S in $\operatorname{aff} S$.

For a subset $X \subseteq \mathcal{Z}$, we define a subset $\operatorname{ev}_X V$ of V^* as follows:

$$\operatorname{ev}_X V := \{\operatorname{ev}_x V \in V^* : x \in X\},$$

where $(\operatorname{ev}_x V)(f) := f(x)$ for all $f \in V$.

Definition 2.13. Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield and V a real vector space. An *$\mathbb{F}$-structure on V* is an $\mathbb{F}$-vector space W such that $V \cong \mathbb{R} \otimes_{\mathbb{F}} W$.

An $\mathbb{F}$-structure W on V gives an $\mathbb{F}$-structure $W^* := \operatorname{Hom}_{\mathbb{F}}(W, \mathbb{F})$ on V^* since $V^* \cong \mathbb{R} \otimes_{\mathbb{F}} W^*$. This isomorphism enables us to view W^* as an $\mathbb{F}$-vector subspace of V^*. A typical example of $\mathbb{F}$-structure comes from a choice of basis of the vector space. Let $\{f_i\}$ be an $\mathbb{R}$-basis of a finite dimensional real vector space V. Then, the $\mathbb{F}$-vector space $W := \bigoplus_i \mathbb{F} f_i$ is an $\mathbb{F}$-structure of V. Moreover, let $\{f_i^*\}$ be the dual basis on V^*, then the $\mathbb{F}$-vector space $W^* = \bigoplus_i \mathbb{F} f_i^*$ is the corresponding $\mathbb{F}$-structure on V^*.

Next, we explore some properties of the centroid of $\mathcal{X}$ and the convex cone of ev_X.

Proposition 2.14. *Let $X \subseteq \mathcal{Z}$ be a subset. Assume that the constant function $1_{\mathcal{Z}}$ on $\mathcal{Z}$ is in the vector space V. The following statements hold.*

- (i) *Let $A := \{\alpha \in V^* : \alpha(1_{\mathcal{Z}}) = 1\}$ be an affine subspace of V^*. Then we have $(\operatorname{cone} \operatorname{ev}_X V) \cap A = \operatorname{conv} \operatorname{ev}_X V$.*
- (ii) *The set X is V-nondegenerate if and only if $\dim \operatorname{cone} \operatorname{ev}_X V = \dim V$. In particular, $\operatorname{int} \operatorname{cone} \operatorname{ev}_X V = \emptyset$ if X is not V-nondegenerate, and $\operatorname{int} \operatorname{cone} \operatorname{ev}_X V = \operatorname{reint} \operatorname{cone} \operatorname{ev}_X V \neq \emptyset$ if X is V-nondegenerate.*
- (iii) *If X is the support of a V-nondegenerate weighted V-design $\mathcal{X}$ on $\mathcal{Z}$, then $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$.*
- (iv) *If $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$, then $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_Y V$ for some finite subset $Y \subseteq X$.*
- (v) *Let W be an $\mathbb{F}$-structure on V and identify W^* with an $\mathbb{F}$-vector subspace of V^*. Assume that X is a finite set and let $p \in V^*$ such that $p(1_{\mathcal{Z}}) = 1$. If $\operatorname{ev}_X V \subseteq W^*$ and $p \in (\operatorname{int} \operatorname{cone} \operatorname{ev}_X V) \cap W^*$, then X is the support of a V-nondegenerate $\mathbb{F}$-weighted levelling space $\mathcal{X}$ such that $\operatorname{centroid}_{\mathcal{X}} V = p$.*
- (vi) *Assume that X is a finite set. If $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$, then X is the support of a finite V-nondegenerate weighted V-design $\mathcal{X}$ on $\mathcal{Z}$.*

Proof. (i) For every $x \in X$, we have $(\operatorname{ev}_x V)(1_{\mathcal{Z}}) = 1_{\mathcal{Z}}(x) = 1$. Therefore, $\sum_{i=1}^n c_i \operatorname{ev}_x V \in A$ if and only if $\sum_{i=1}^n c_i = 1$, hence the result.
(ii) Since V is finite dimensional, $\dim \operatorname{cone} \operatorname{ev}_X V = \dim V|_X$, hence the result.
(iii) We have

$$\begin{aligned}
\operatorname{centroid}_{\mathcal{Z}} V &= \operatorname{centroid}_{\mathcal{X}} V && \mathcal{X} \text{ is a } V\text{-design on } \mathcal{Z} \\
&\in \operatorname{reint} \operatorname{conv} \operatorname{ev}_X V && \text{[SZ84, Lemma 3.1] applied to } \mathcal{X} \\
&\subseteq \operatorname{reint} \operatorname{cone} \operatorname{ev}_X V && \text{(i)} \\
&= \operatorname{int} \operatorname{cone} \operatorname{ev}_X V. && \text{(ii)}
\end{aligned}$$

(iv) Let $p' := p/p(1_{\mathbb{Z}}) \in A$. By (i), p' is in the interior of $\operatorname{conv}\operatorname{ev}_X V$ in A. Applying Steinitz's theorem [GW93, Part 2.1, Theorem 10.3], we find a finite set $Y \subseteq X$ such that p' is in the interior of $\operatorname{conv}\operatorname{ev}_Y V$ in A. Therefore, by (i) again, $p' \in \operatorname{int}\operatorname{cone}\operatorname{ev}_Y V$, hence $p \in \operatorname{int}\operatorname{cone}\operatorname{ev}_Y V$.
(v) Since $p \in \operatorname{int}\operatorname{cone}\operatorname{ev}_X V$, by (ii), X is V-nondegenerate. By (i) and (ii), $p \in (\operatorname{int}\operatorname{cone}\operatorname{ev}_X V) \cap A = \operatorname{reint}\operatorname{conv}\operatorname{ev}_X V$, hence p can be written as a positive convex linear combination of points in $\operatorname{ev}_X V$, namely $p = \sum_{x\in X} c_x \operatorname{ev}_x V$ for some $c_x \in (0,1)$ such that $\sum_{x\in X} c_x = 1$. Since p and $\operatorname{ev}_X V$ are both in the $\mathbb{F}$-structure W^*, we may in addition assume that $c_x \in (0,1)_{\mathbb{F}}$ for every $x \in X$. Then the levelling space $\mathcal{X} := (X, \mu_X)$ where $\mu_X(x) := c_x$ satisfies the requirement $\operatorname{centroid}_{\mathcal{X}} V = p$.
(vi) It is clear that $\operatorname{centroid}_{\mathcal{Z}} 1_{\mathcal{Z}} = 1$. The result follows by applying (v) to $\mathbb{F} = \mathbb{R}$, $W = V$ and $p = \operatorname{centroid}_{\mathcal{Z}} V$. □

Proposition 2.15. *Assume that $1_{\mathcal{Z}} \in V$. Then, there exists a finite V-nondegenerate weighted V-design on $\mathcal{Z}$.*

Proof. Applying Proposition 2.14(iii) to the V-nondegenerate weighted V-design $\mathcal{Z}$ itself on $\mathcal{Z}$, we get $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int}\operatorname{cone}\operatorname{ev}_Z V$. Then by Proposition 2.14(iv), there exists a finite subset $X \subseteq Z$ such that $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int}\operatorname{cone}\operatorname{ev}_X V$. Therefore, by Proposition 2.14(vi), X is the support of a finite V-nondegenerate weighted V-design. □

Lemma 2.16. *Assume that $1_{\mathcal{Z}} \in V$. Let $X \subseteq \mathcal{Z}$ be the support of a finite V-nondegenerate weighted V-design on $\mathcal{Z}$, and $S \subseteq \mathcal{Z}$ a finite subset. Then, $X \cup S$ is the support of a finite V-nondegenerate weighted V-design on $\mathcal{Z}$.*

Proof. By Proposition 2.14(iii), $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int}\operatorname{cone}\operatorname{ev}_X V \subseteq \operatorname{int}\operatorname{cone}\operatorname{ev}_{X\cup S} V$. The result follows from Proposition 2.14(vi). □

Lemma 2.17. *Assume that $1_{\mathcal{Z}} \in V$. Let $\mathcal{X} \subseteq \mathcal{Z}$ be a finite V-nondegenerate levelling space. For each point $x \in \mathcal{X}$, let $(x^{(i)} \in \mathcal{Z})_{i\in\mathbb{N}}$ be a sequence of points such that $\lim_{i\to\infty} \operatorname{ev}_{x^{(i)}} V = \operatorname{ev}_x V$ in V^*. Then, for all sufficiently large i, $X^{(i)} := \{x^{(i)} : x \in \mathcal{X}\}$ is the support of a finite V-nondegenerate levelling space $\mathcal{X}^{(i)}$ that is V-equivalent to $\mathcal{X}$.*

Proof. We equip V^* with a metric that is compatible with the topology of V^*, say the metric induced from $\mathbb{R}^{\dim V}$. It is easy to see that $\operatorname{ev}_{X^{(i)}} V$ converges to $\operatorname{ev}_X V$ with respect to Hausdorff distance in V^*, hence $\operatorname{cone}\operatorname{ev}_{X^{(i)}} V$ converges to $\operatorname{cone}\operatorname{ev}_X V$ with respect to Hausdorff distance on compact supports in V^*. Since $\mathcal{X}$ is V-nondegenerate, for all sufficiently large i, by Proposition 2.14(ii), $\dim\operatorname{cone}\operatorname{ev}_{X^{(i)}} V = \dim\operatorname{cone}\operatorname{ev}_X V = \dim V$ which shows that $X^{(i)}$ is V-nondegenerate, hence $\operatorname{int}\operatorname{cone}\operatorname{ev}_{X^{(i)}} V$ converges to $\operatorname{int}\operatorname{cone}\operatorname{ev}_X V$ with respect to Hausdorff distance on compact supports in V^*. Applying Proposition 2.14(iii) to the V-nondegenerate weighted V-design $\mathcal{X}$ on $\mathcal{Z}$, we get $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int}\operatorname{cone}\operatorname{ev}_X V$. Therefore, for all sufficiently large i, $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int}\operatorname{cone}\operatorname{ev}_{X^{(i)}} V$, and the result follows from Proposition 2.14(vi). □

3. Weighted rational designs

In this section, we first define rational designs and rational weighted designs in Euclidean spaces. We adopt Definition 3.1 for designs in the remainder of the paper unless stated explicitly otherwise. Then, we show the necessity of Conditions 1.3 and 1.4 in Propositions 3.3 and 3.5, respectively. Conversely, in Lemmas 3.4 and 3.7 we prove that these conditions guarantee the existence of certain weighted rational designs. Both Lemmas 3.4 and 3.7 are used in the proof of Theorem 7.1.

3.1. **Designs in Euclidean spaces.** Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space. The algebra of all *polynomials on $\mathcal{Z}$* is defined to be

$$\mathcal{P}[\mathcal{Z}] := \mathbb{R}[x_1, \ldots, x_d]/I(\mathcal{Z}),$$

where $I(\mathcal{Z})$ is the ideal of polynomials in $\mathbb{R}[x_1, \ldots, x_d]$ vanishing on $\mathcal{Z}$. For an ideal I in $\mathbb{R}[x_1, \ldots, x_d]$, let $I_{\leq t}$ be the vector subspace of I consisting of polynomials of degree bounded above by t. The algebra $\mathcal{P}[\mathcal{Z}]$ admits a filtration of vector spaces:

$$0 \subseteq \mathcal{P}^0[\mathcal{Z}] \subseteq \mathcal{P}^1[\mathcal{Z}] \subseteq \cdots,$$

where

$$\mathcal{P}^t[\mathcal{Z}] := \mathbb{R}[x_1, \ldots, x_d]_{\leq t} / I(\mathcal{Z})_{\leq t}$$

is a vector subspace of $\mathcal{P}[\mathcal{Z}]$. In other words, $\mathcal{P}^t[\mathcal{Z}]$ is the vector subspace of $\mathcal{P}[\mathcal{Z}]$ generated by polynomials of degree at most t.

In particular, on the real unit d-sphere $\mathcal{S}^d$,

$$\mathcal{P}^t[\mathcal{S}^d] = \mathbb{R}[x_0, \ldots, x_d]_{\leq t} / (x_0^2 + \cdots + x_d^2)_{\leq t},$$

and on the unit open interval $\mathcal{I}$,

$$\mathcal{P}^t[\mathcal{I}] = \mathbb{R}[x]_{\leq t}.$$

Definition 3.1. Suppose that all polynomials on $\mathcal{Z}$ are integrable on $\mathcal{Z}$. A *t-design (resp. weighted t-design) $\mathcal{X}$ on $\mathcal{Z}$* is a $\mathcal{P}^t[\mathcal{Z}]$-design (resp. weighted $\mathcal{P}^t[\mathcal{Z}]$-design) on $\mathcal{Z}$ (as introduced in Definition 2.11). A (weighted) t-design $\mathcal{X}$ on $\mathcal{Z}$ is called *rational* if it consists of rational points, namely $\mathcal{X} \subseteq \mathbb{Q}^d$.

Remark 3.2. Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space. Let $\mathcal{X} = (X, \mu_X) \subseteq \mathcal{Z}$ be a levelling space such that μ_X is the counting measure. Then, the levelling space $\mathcal{X}$ is a t-design on $\mathcal{Z}$ in the sense of Definition 3.1 if and only if the support X is a t-design on $\mathcal{Z}$ in the sense of Definition 1.1.

We say that there are *enough designs on $\mathcal{Z}$* satisfying some given additional properties provided that there are t-designs on $\mathcal{Z}$ satisfying the properties for each natural number t.

3.2. **Necessity and sufficiency of Condition 1.3.** Recall that in Definition 2.7 we define the *centroid of f on $\mathcal{X}$* of a function f to be $\operatorname{centroid}_{\mathcal{X}} f := \frac{1}{\mu_X(X)} \int_X f \, \mathrm{d}\, \mu_X$.

Proposition 3.3. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a bounded levelling space, and $Y \subseteq \mathcal{Z}$ a subset. If there are enough weighted designs on $\mathcal{Z}$ consisting of points in Y, then Y is dense in $\mathcal{Z}$. In particular, Condition 1.3 is a necessary condition for $\mathcal{Z}$ to have enough rational designs.*

Proof. Let p be an arbitrary point in $\mathcal{Z}$. For each $e \in \mathbb{N}$, let $\mathcal{X}_{2e}$ be a weighted $2e$-design on $\mathcal{Z}$ consisting of points in Y. Since $\mathcal{Z}$ is bounded, there exists a real number d that is larger than the diameter of $\mathcal{Z}$. Then, the quadratic polynomial f_p on $\mathcal{Z}$ given by

$$f_p(x) := 1 - \frac{1}{d^2} \|x - p\|_2^2$$

has the property that $f_p(\mathcal{Z}) \subseteq (0, 1]$ and f_p achieves maximal value 1 at the point p. For $\epsilon \in (0, 1]$, denote by p_ϵ the preimage $f_p^{-1}((1 - \epsilon, 1])$, which is an open set in $\mathcal{Z}$ and has positive measure.

Suppose that Y is not dense around p. Let $\delta := 1 - \sup f_p(Y) > 0$, and let $\epsilon \in (0, \delta)$ be a fixed number. Since f_p^e is of degree $2e$ and $\mathcal{X}_{2e}$ is a weighted $2e$-design,

$$(1 - \delta)^e \geq \operatorname{centroid}_{\mathcal{X}_{2e}} f_p^e = \operatorname{centroid}_{\mathcal{Z}} f_p^e \geq \frac{\mu_Z(p_\epsilon)}{\mu_Z(Z)} (1 - \epsilon)^e,$$

where $\mu_Z(p_\epsilon) > 0$. Taking $e \to \infty$, we get a contradiction. Therefore, Y is dense around arbitrary point $p \in \mathcal{Z}$. $\square$

Lemma 3.4. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space where Condition 1.3 holds and polynomials are integrable on $\mathcal{Z}$. Then, for every $t \in \mathbb{N}$, there exists a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design on $\mathcal{Z}$.*

Proof. Applying Proposition 2.15 with $V = \mathcal{P}^t[\mathcal{Z}]$, we get a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted t-design $\mathcal{X}$ on $\mathcal{Z}$. For each point $x \in \mathcal{X}$, we pick a sequence of rational points $(x^{(i)} \in \mathcal{Z} \cap \mathbb{Q}^d)_{i \in \mathbb{N}}$ whose limit is x. By Lemma 2.17, we know that for some sufficiently large i, $X^{(i)} := \{x^{(i)} : x \in \mathcal{X}\}$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate levelling space $\mathcal{X}^{(i)}$ that is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\mathcal{X}$. Then, $\mathcal{X}^{(i)}$ is a desired design. □

3.3. **Necessity and sufficiency of Condition 1.4.** Let $k \subseteq \mathbb{R}$ be a subset. Recall that in Definition 2.2, we say that a levelling space $\mathcal{X}$ is *k-weighted* if μ_X takes values in k.

Proposition 3.5. *Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield, and let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space. If there are enough finite $\mathbb{F}$-weighted rational designs on $\mathcal{Z}$, then* $\operatorname{centroid}_{\mathcal{Z}} f \in \mathbb{F}$ *for every monic monomial f in $\mathcal{P}[\mathcal{Z}]$. In particular, Condition 1.4 is a necessary condition for having enough $\mathbb{Q}$-weighted rational designs on $\mathcal{Z}$.*

Proof. Let t be a natural number and $\mathcal{X}$ a finite $\mathbb{F}$-weighted rational t-design on $\mathcal{Z}$. By the definition of weighted designs, for every monic monomial f in $\mathcal{P}^t[\mathcal{Z}]$,

$$\operatorname{centroid}_{\mathcal{Z}} f = \operatorname{centroid}_{\mathcal{X}} f = \frac{1}{\mu_X(X)} \sum_{x \in \mathcal{X}} \mu_X(x) f(x),$$

which is an $\mathbb{F}$-linear combination of rational numbers $\{f(x) : x \in \mathcal{X}\}$. Therefore, $\operatorname{centroid}_{\mathcal{Z}} f \in \mathbb{F}$. Since the choice of t is arbitrary, the result holds for all monic monomials $f \in \bigcup_{t \in \mathbb{N}} \mathcal{P}^t[\mathcal{Z}] = \mathcal{P}[\mathcal{Z}]$. □

Lemma 3.6. *Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield and $\mathcal{Z} \subseteq \mathbb{R}^d$ a levelling space. Suppose that $X \subseteq \mathcal{Z} \cap \mathbb{Q}^d$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted levelling space $\mathcal{X}$ such that* $\operatorname{centroid}_{\mathcal{X}} f \in \mathbb{F}$ *for all monic monomials $f \in \mathcal{P}^t[\mathcal{Z}]$. Then, X is also the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate $\mathbb{F}$-weighted levelling space $\widetilde{\mathcal{X}}$ that is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\mathcal{X}$. Moreover, when $\mathbb{F} = \mathbb{Q}$, $\widetilde{\mathcal{X}}$ can be taken to be integer-weighted.*

Proof. Choose a basis of $\mathcal{P}^t[\mathcal{Z}]$ consisting of monic monomials, and let $\mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]$ be the $\mathbb{F}$-vector space generated by the basis. Let $p := \operatorname{centroid}_{\mathcal{X}} \mathcal{P}^t[\mathcal{Z}] \in \mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]^*$. By Proposition 2.14(iii), $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X \mathcal{P}^t[\mathcal{Z}]$. Since X consists of rational points, $\operatorname{ev}_X \mathcal{P}^t[\mathcal{Z}] \subseteq \mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]^*$. Applying Proposition 2.14(v) to the $\mathbb{F}$-structure $\mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]$ of $\mathcal{P}^t[\mathcal{Z}]$, point p and finite set X, we get a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate $\mathbb{F}$-weighted levelling space $\widetilde{\mathcal{X}}$ with support X that is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\mathcal{X}$. When $\mathbb{F} = \mathbb{Q}$, a suitable integer scalar $n\widetilde{\mathcal{X}}$ is an integer-weighted levelling space, which is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\widetilde{\mathcal{X}}$ by Lemma 2.9. □

Lemma 3.7. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space satisfying Condition 1.4. Suppose that $\mathcal{X} \subseteq \mathcal{Z}$ is a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design on $\mathcal{Z}$ with support X. Then, X is the support of a $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate integer-weighted rational t-design $\widetilde{\mathcal{X}}$ on $\mathcal{Z}$.*

Proof. Since $\operatorname{centroid}_{\mathcal{X}} \mathcal{P}^t[\mathcal{Z}] = \operatorname{centroid}_{\mathcal{Z}} \mathcal{P}^t[\mathcal{Z}]$, applying Lemma 3.6 to $\mathbb{F} = \mathbb{Q}$ and $\mathcal{X}$, we get a desired t-design $\widetilde{\mathcal{X}}$. □

The following proposition is stronger than Lemma 3.4. It will not be used in the remainder of this article as Lemma 3.4 suffices for our purpose.

Proposition 3.8. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space satisfying Conditions 1.3 and 1.4. Assume that $\mathcal{Z}$ is not finite. Then, for every $t \in \mathbb{N}$ and every sufficiently large $n \in \mathbb{N}$, there exists a $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate integer-weighted rational t-design $\mathcal{X}$ on $\mathcal{Z}$ of size n.*

Proof. By Lemma 3.4, there exists a finite weighted $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate rational t-design $\mathcal{X}$ on $\mathcal{Z}$. Let $S \subseteq \mathcal{Z} \cap \mathbb{Q}^d$ be a finite subset such that $|X \cup S| = n$. By Lemma 2.16, $X \cup S$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design of size n. Then, according to Lemma 3.7, $X \cup S$ is the support of a desired design. □

4. Total measures of integer-weighted designs

The main result in this section is Theorem 4.5, which shows that for all sufficiently large total measures, we can find an integer-weighted levelling space that is equivalent to a fixed levelling space. Theorem 4.5 is used in the proof of Theorem 7.1.

4.1. **Vandermonde matrix and its p-adic valuation.** For a prime number p, let ν_p be the p-adic valuation. In other words, for $n \in \mathbb{Z}$, $\nu_p(n) := \sup\{v \in \mathbb{N} : p^v | n\}$, and for $a, b \in \mathbb{Z}$, $\nu_p(a/b) := \nu_p(a) - \nu_p(b)$. For a matrix $\mathbf{A}$ over $\mathbb{Q}$, let $\nu_p(\mathbf{A}) := \min\{\nu_p(a) : \text{ entry } a \text{ of } \mathbf{A}\}$. For two matrices $\mathbf{A}$ and $\mathbf{A}'$ over $\mathbb{Q}$, we have $\nu_p(\mathbf{A}\mathbf{A}') \geq \nu_p(\mathbf{A}) + \nu_p(\mathbf{A}')$ whenever the sizes of $\mathbf{A}$ and $\mathbf{A}'$ are compatible.

Definition 4.1. Let t and n be two natural numbers, and let $\mathbf{a} := (a_1, \ldots, a_n)$ be a tuple of rational numbers. The *t-th Vandermonde matrix of* $\mathbf{a}$ is the matrix $\mathbf{A}$ whose rows are indexed by $[0, t]_{\mathbb{Z}}$, columns indexed by $[1, n]_{\mathbb{Z}}$ and (i, j)-th entry defined to be $\mathbf{A}_{i,j} := a_j^i$, where $i \in [0, t]_{\mathbb{Z}}$ and $j \in [1, n]_{\mathbb{Z}}$.

Lemma 4.2. *Let n be a natural number, $\mathbf{a} := (a_1, \ldots, a_n)$ a tuple of rational numbers and $\mathbf{A}$ the $(n-1)$-th Vandermonde matrix of $\mathbf{a}$, which is a square matrix. Suppose that p is a prime number such that $\nu_p(a_i)$'s are distinct negative integers where i runs over $[1, n]_{\mathbb{Z}}$. Then, for $i \in [1, n]_{\mathbb{Z}}$ and $j \in [0, n-1]_{\mathbb{Z}}$, we have $\nu_p((\mathbf{A}^{-1})_{i,j}) \geq -j\nu_p(a_i)$. In particular, $\nu_p((\mathbf{A}^{-1})_{i,j}) \geq j\nu_p(1/\mathbf{a})$, where $1/\mathbf{a} := (1/a_1, \ldots, 1/a_n)$.*

Proof. It is well known that the (i, j)-th entry of $\mathbf{A}^{-1}$ is

$$(\mathbf{A}^{-1})_{i,j} = (-1)^j \frac{e_{n-j-1}(a_1, \ldots, \hat{a}_i, \ldots, a_n)}{\displaystyle\prod_{\substack{k \in [1,n]_{\mathbb{Z}} \\ k \neq i}} (a_k - a_i)},$$

where $e_{n-j-1}(a_1, \ldots, \hat{a}_i, \ldots, a_n)$ is the $(n-j-1)$-th elementary symmetric polynomial. Since all $\nu_p(a_i)$'s are distinct, it is straightforward to calculate the p-adic valuation of numerators and denominators of $(\mathbf{A}^{-1})_{i,j}$, and verify that $\nu_p((\mathbf{A}^{-1})_{i,j}) \geq -j\nu_p(a_i)$. □

Lemma 4.3. *Let t and n be two natural numbers such that $n \geq t+1$, $\mathbf{a} := (a_1, \ldots, a_n)$ a tuple of positive rational numbers, $\mathbf{A}$ the t-th Vandermonde matrix of $\mathbf{a}$, and $\mathbf{b} := (b_0, \ldots, b_t) \in \mathbb{Q}^{t+1}$ a column vector. Suppose that p is a prime number such that $\nu_p(a_i)$'s are distinct negative integers where i runs over $[1, n]_{\mathbb{Z}}$. Assume that the linear system*

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{4.1}$$

has a positive real solution $\mathbf{x} \in \mathbb{R}^n_{>0}$. Then, Eq. (4.1) has a positive rational solution $\mathbf{y} \in \mathbb{Q}^n_{>0}$ such that $\nu_p(\mathbf{y}) \geq \min\{j\nu_p(1/\mathbf{a}) + \nu_p(b_j) : j \in [0, t]_{\mathbb{Z}}\}$. In particular, if $\nu_p(1/\mathbf{a})$ is sufficiently large, then $\nu_p(\mathbf{y}) \geq \nu_p(b_0)$.

Proof. For a subset $S \subseteq [1, n]_{\mathbb{Z}}$, denote its complement by $\overline{S} := [1, n]_{\mathbb{Z}} \setminus S$, and let $\cdot|_S$ and $\cdot|_{\overline{S}}$ be the restriction maps to S and $\overline{S}$, respectively.

Since Eq. (4.1) has a positive real solution, there exists a positive rational number c such that $\nu_p(c)$ is sufficiently large and that the system

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq c\mathbf{1}, \end{cases} \tag{4.2}$$

has a real solution with $\mathbf{x} > c\mathbf{1}$. Clearly, the dimension of the solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $n - t - 1$. Moreover, due to the existence of a solution $\mathbf{x} > c\mathbf{1}$, the convex polytope defined by Eq. (4.2) also has dimension $n - t - 1$. Let $\mathbf{y}$ be an extreme point of this convex polytope. We then know that $\mathbf{y}$ can be uniquely determined by the intersection of n hyperplanes, $t + 1$ of which are given by $\mathbf{A}\mathbf{x} = \mathbf{b}$, and the other $n - t - 1$ are of the form $x_i = c$ where $i \in \overline{S}$ for some subset $S \subseteq [1, n]_{\mathbb{Z}}$ of size $t + 1$.

It is then easy to see from Eq. (4.2) that $\mathbf{A}|_S \cdot \mathbf{y}|_S + \mathbf{A}|_{\overline{S}} \cdot \mathbf{y}|_{\overline{S}} = \mathbf{b}$. Since $\mathbf{y}$ is an extreme point, $\mathbf{A}|_S$ is an invertible Vandermonde matrix and $\mathbf{y}$ is a rational point whose coordinates are given by

$$\begin{cases} \mathbf{y}|_S = \mathbf{A}|_S^{-1} \cdot (\mathbf{b} - \mathbf{A}|_{\overline{S}} \cdot \mathbf{y}|_{\overline{S}}), \\ \mathbf{y}|_{\overline{S}} = c\mathbf{1}. \end{cases} \tag{4.3}$$

Thus,

$$\begin{aligned} \nu_p(\mathbf{y}) &= \min\{\nu_p(\mathbf{y}|_S), \nu_p(\mathbf{y}|_{\overline{S}})\} \\ &\geq \min\big\{\nu_p(\mathbf{A}|_S^{-1} \cdot \mathbf{b}), \\ &\quad \nu_p(\mathbf{A}|_S^{-1}) + \nu_p(\mathbf{A}|_{\overline{S}}) + \nu_p(c), \nu_p(c)\big\} && \text{Eq. (4.3)} \\ &= \nu_p(\mathbf{A}|_S^{-1} \cdot \mathbf{b}) && \nu_p(c) \text{ is sufficiently large} \\ &\geq \min\{j\nu_p(1/\mathbf{a}|_S) + \nu_p(b_j) :\ j \in [0,t]_{\mathbb{Z}}\}, && \text{Lemma 4.2 applied to } \mathbf{A}|_S \\ &\geq \min\{j\nu_p(1/\mathbf{a}) + \nu_p(b_j) :\ j \in [0,t]_{\mathbb{Z}}\}, && \nu_p(1/\mathbf{a}|_S) \geq \nu_p(1/\mathbf{a}) \end{aligned}$$

which completes the proof. □

4.2. **Realization of total measures.**

Proposition 4.4. *Let t be a natural number and let $\mathcal{X} \subseteq \mathcal{I} \cap \mathbb{Q}$ be a finite $\mathcal{P}^t[\mathcal{I}]$-nondegenerate levelling space. For every prime number p, there exists an integer-weighted levelling space $\mathcal{X}_p \subseteq \mathcal{I} \cap \mathbb{Q}$ such that $\mathcal{X}_p$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$ and the total measure of $\mathcal{X}_p$ is not divisible by p.*

Proof. Let $\mathbf{b} := (b_j :\ j \in [0,t]_{\mathbb{Z}})$, where $b_j := \operatorname{centroid}_{\mathcal{X}} f_j$ and $f_j(x) := x^j$. It is clear that $b_0 = 1$. For each $x \in \mathcal{X}$, we choose a rational point $a_x \in \mathcal{I} \cap \mathbb{Q}$ sufficiently close to x while $\nu_p(1/a_x)$'s are sufficiently large distinct natural numbers. Let $\mathbf{a} := \{a_x :\ x \in \mathcal{X}\}$ and $\mathbf{A}$ the t-th Vandermonde matrix of $\mathbf{a}$.

According to Lemma 2.17, $\mathbf{a}$ is the support of a finite levelling space $\mathcal{A}$ that is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$. So, Eq. (4.1) has a positive real solution $\mathbf{u} := (\mu_A(a_x)/\mu_A(A) :\ x \in \mathcal{X})$. Then, by Lemma 4.3, Eq. (4.1) also has a positive rational solution $\mathbf{w} = (w_x :\ x \in \mathcal{X})$ such that $\nu_p(\mathbf{w}) \geq \nu_p(b_0) = 0$. The solution $\mathbf{w}$ gives us a rational-weighted levelling space $\mathcal{W}$ with support $\mathbf{a}$, measure $\mu_W(a_x) = w_x$ for all $x \in \mathcal{X}$ and total measure $\sum_{x\in\mathcal{X}} w_x = b_0 = 1$. The levelling space $\mathcal{W}$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{A}$ since $\mathbf{Aw} = \mathbf{Au}$.

Let m be the least common multiple of the denominators of the rational coordinates of $\mathbf{w}$. Since $\nu_p(\mathbf{w}) \geq 0$, m is not divisible by p. Therefore, $m\mathcal{W}$ is an integer-weighted levelling space, which is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{W}$ by Lemma 2.9, with total measure m, which is not divisible by p. □

Theorem 4.5. *Let t be a natural number and let $\mathcal{X} \subseteq \mathcal{I} \cap \mathbb{Q}$ be a $\mathcal{P}^t[\mathcal{I}]$-nondegenerate integer-weighted levelling space. Then, there exists a natural number n_0 such that for every natural number $n \geq n_0$, there exists an integer-weighted levelling space $\mathcal{X}_n \subseteq \mathcal{I} \cap \mathbb{Q}$ with total measure n such that $\mathcal{X}_n$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$.*

Proof. Let P be the set of all prime factors of $\mu_X(X)$ where X is the support of $\mathcal{X}$. For each $p \in P$, we apply Proposition 4.4 and get an integer-weighted levelling space $\mathcal{X}_p \subseteq \mathcal{I} \cap \mathbb{Q}$ with support X_p that is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$ such that $\mu_{X_p}(X_p)$ is not divisible by p. Therefore, the additive semigroup generated by $\mu_X(X)$ and $\mu_{X_p}(X_p)$ for all $p \in P$, which is a numerical semigroup, contains all sufficiently large natural numbers. Let n be an arbitrary sufficiently large natural number. Then, it can be written as a finite linear combination

$$n = c_0\mu_X(X) + \sum_{p\in P} c_p \mu_{X_p}(X_p)$$

for some natural numbers c_0 and c_p. By Lemma 2.9,

$$\mathcal{X}_n := c_0\,\mathcal{X} + \sum_{p\in P} c_p\,\mathcal{X}_p$$

is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$. Moreover, $\mathcal{X}_n$ is integer-weighted and has total measure n. $\square$

5. Hilbert-Kamke problem

We present in this section some results on the Hilbert-Kamke problem. Our main result in this section is Theorem 5.3, which is used in the proof of Theorem 7.1.

Let t and n be fixed natural numbers and let $\mathbf{c} := (c_1, \dots, c_t)$ be a fixed tuple of rational numbers. Consider the Hilbert-Kamke type system of t equations in n variables x_i's:

$$\sum_{i=1}^{n} x_i^k = c_k, \quad k \in [1,t]_{\mathbb{Z}}. \tag{5.1}$$

For a positive integer P, denote by $\mathrm{J}(t,n;\mathbf{c};P)$ the set of all solutions of Eq. (5.1) such that $x_i \in (0,1)_{P^{-1}\mathbb{Z}}$ for all $i \in [1,n]_{\mathbb{Z}}$, where $P^{-1}\mathbb{Z} := \{P^{-1}q : q \in \mathbb{Z}\}$ is a fractional ideal over $\mathbb{Z}$.

Remark 5.1. Let $P_{\mathbf{c}}$ be the smallest positive integer P such that $c_k \in P^{-k}\mathbb{Z}$ for every integer $k \in [1,t]_{\mathbb{Z}}$. Then $\mathrm{J}(t,n;\mathbf{c};P) = \emptyset$ unless P is a positive integer multiple of $P_{\mathbf{c}}$.

The study of $\mathrm{J}(t,n;\mathbf{c};P)$ is a variation of the famous Hilbert-Kamke problem, which has been being addressed over the last several decades [Mar53, Ark85, Woo12]. For our purpose, we need to show the existence of solutions in $\mathrm{J}(t,n;\mathbf{c};P)$ satisfying some additional properties, which is obtained through an asymptotic formula for the size of $\mathrm{J}(t,n;\mathbf{c};P)$ when P goes to infinity in $P_{\mathbf{c}}\mathbb{Z}_{>0}$. We reformulate below in Theorem 5.2 the asymptotic results in [Ark85] to suit our purpose.

We use Vinogradov's notation for asymptotic formulas in this paper. Let f and g be two functions on a domain $D \subseteq \mathbb{R}$, and let $y_1, \dots, y_n$ be some objects. Denote by $f \ll_{y_1,\dots,y_n} g$ the fact that there exists a positive constant $c_{y_1,\dots,y_n}$ that only depends on $y_1, \dots, y_n$ such that $|f(x)| \le c_{y_1,\dots,y_n}|g(x)|$ for all sufficiently large $x \in D$. We write $f \gg_{y_1,\dots,y_n} g$ if $g \ll_{y_1,\dots,y_n} f$, and write $f \asymp_{y_1,\dots,y_n} g$ if $f \ll_{y_1,\dots,y_n} g \ll_{y_1,\dots,y_n} f$.

Theorem 5.2. *Let t, n, $\mathbf{c}$ and $P_{\mathbf{c}}$ be fixed as above. Suppose that $n \ge 3t^2 2^t - t$ and consider the domain $D := \{P \in P_{\mathbf{c}}\mathbb{Z}_{>0} : P \ge n^{10}\}$. Then there exist real-valued functions σ, γ and θ of P such that the following statements hold.*

(i) *On the domain D,*

$$|\mathrm{J}(t,n;\mathbf{c};P)| = \sigma \cdot \gamma \cdot (P-1)^{n-t(t+1)/2} + \theta \cdot n^{30n^3}(P-1)^{n-t(t+1)/2-1/30(2+\log t)}.$$

(ii) *For all $P \ge 1$,*

$$|\mathrm{J}(t,n;\mathbf{c};P)| \le n^{30n^3}(P-1)^{n-t(t+1)/2}.$$

(iii) *On the domain D,*

$$|\theta| < 1.$$

(iv) *If there exists a p-adic solution $\mathbf{y} \in (P^{-1}\mathbb{Z}_p)^n \subseteq \mathbb{Q}_p^n$ to Eq. (5.1) for each prime number p, then on the domain D,*

$$\sigma \ge n^{-20n^4 2^n}.$$

(v) *If there exists a real solution $\mathbf{y} = (y_1, \dots, y_n) \in [P^{-1}, 1-P^{-1}]^n$ to Eq. (5.1), then on the domain D,*

$$2^{2t(t-n)}n^{(t-n)}t^{-n-t}(\Delta\mathbf{y})^{t(n-t)} \le \gamma \le 2^{2t^2}n^{2t}t^{n-2t}(t+1)^{3t-n},$$

where

$$\Delta\mathbf{y} := \max_{z_0,\dots,z_{t+1}\in\{0,y_1,\dots,y_n,1\}} \left(\min_{0\le i<j\le t+1} |z_i - z_j|\right).$$

(vi) *If there exists a rational solution* $\mathbf{y} = (y_1, \ldots, y_n) \in (0,1)^n_{\mathbb{Q}}$ *to Eq.* (5.1) *such that the number of distinct elements of* $y_1, \ldots, y_n$ *is at least* t*, then*

$$|\mathrm{J}(t, n; \mathbf{c}; P)| \asymp_{t,n,\mathbf{y}} P^{n-t(t+1)/2}$$

as $P \to \infty$ *in the domain* D.

Proof. Although [Ark85] assumes that $t \geq 3$, the proof there also works for $t \in \{0, 1, 2\}$. So, when we refer to theorems in [Ark85], such restriction on t will be omitted.

Let $N_k := c_k P^k$ for all $k \in [1, t]_{\mathbb{Z}}$. It is clear that there exists a bijection sending a solution $(y_1, \ldots, y_n) \in (0,1)^n_{P^{-1}\mathbb{Z}}$ of Eq. (5.1) to a solution $(Py_1, \ldots, Py_n) \in [1, P-1]^n_{\mathbb{Z}}$ of

$$\sum_{i=1}^{n} x_i^k = N_k, \quad k \in [1, t]_{\mathbb{Z}}.$$

Now we consider the counting function $J(t, n; N_1, \ldots, N_t; P-1)$ appearing in [Ark85, Theorem 1]. Then, by definition, we have

$$|\mathrm{J}(t, n; \mathbf{c}; P)| = J(t, n; N_1, \ldots, N_t; P-1).$$

Let

$$\sigma := \sigma(t, n; P-1)$$

be the sum of the singular series defined in [Ark85, Theorem 1] and

$$\gamma := \gamma(t, n; N_1, \ldots N_t; P-1)$$

be the value of the singular integral defined in [Ark85, Theorem 1].

(i), (ii), (iii): The result follows from [Ark85, Theorem 1].

(iv): The result follows from [Ark85, Theorem 3, 4].

(v): The result follows from [Ark85, Theorem 5].

(vi): As discussed in Remark 5.1, the rational solution $\mathbf{y}$ must be in $(0,1)^n_{P^{-1}\mathbb{Z}}$ for some $P \in D$. Then, the solution $\mathbf{y}$ is automatically a p-adic solution in $(P^{-1}\mathbb{Z}_p)^n$ for each prime p and a real solution in $[P^{-1}, 1-P^{-1}]^n$. Since there exist at least t distinct elements in $y_1, \ldots, y_n$, we know that $\Delta\mathbf{y} > 0$. Therefore by (iv) and (v), $\sigma, \gamma \gg_{t,n,\mathbf{y}} 1$. Combining the statements (i), (ii) and (iii) yields (vi). ∎

Using the asymptotic formula from Theorem 5.2, we count in Theorem 5.3 the number of solutions such that x_i's are not the same in certain sense, and formulate the result in the language of levelling spaces.

Theorem 5.3. *Let* t *be a natural number, and let* $\mathcal{X} \subseteq \mathcal{I} \cap \mathbb{Q}$ *be an integer-weighted levelling space with total measure* n*. Suppose that* $|\,\mathcal{X}\,| \geq t$ *and* $n \geq 3t^2 2^t - t$*. Let* $p \in \mathbb{Q}[x]^d$ *be a nonconstant polynomial map* $\mathbb{Q} \to \mathbb{Q}^d$ *for some natural number* d*. For a positive integer* P*, denote by* $\mathrm{J}_p(t; \mathcal{X}; P)$ *and* $\widetilde{\mathrm{J}}_p(t; \mathcal{X}; P)$ *the set of all integer-weighted levelling spaces* $\mathcal{Y} = (Y, \mu_Y)$*, such that* μ_Y *is the counting measure, and* μ_Y *is not the counting measure, respectively, and satisfying the conditions:*

(i) $\mathcal{Y} \subseteq \mathcal{I} \cap P^{-1}\mathbb{Z}$*;*
(ii) $\mu_Y(Y) = n$*;*
(iii) $\mathcal{Y}$ *is* $\mathcal{P}^t[\mathcal{I}]$*-equivalent to* $\mathcal{X}$*;*
(iv) p *is injective on* Y*.*

Then,

$$|\mathrm{J}_p(t; \mathcal{X}; P)| \asymp_{t,\mathcal{X},p} P^{\mu_X(\mathcal{X})-t(t+1)/2}, \tag{5.2}$$

and

$$|\widetilde{\mathrm{J}}_p(t; \mathcal{X}; P)| \ll_{t,\mathcal{X},p} P^{\mu_X(\mathcal{X})-1-t(t+1)/2}, \tag{5.3}$$

as $P \to \infty$ in the domain $P_0\,\mathbb{Z}_{>0}$ for some positive integer P_0 which depends only on t and $\mathcal{X}$. For $P \notin P_0\,\mathbb{Z}_{>0}$, both $\mathrm{J}_p(t;\mathcal{X};P)$ and $\widetilde{\mathrm{J}}_p(t;\mathcal{X};P)$ are empty.

Proof. The set of monic monomials $\{f_k :\ k \in [0,t]_{\mathbb{Z}}\}$, where $f_k(x) := x^k$, form a basis of $\mathcal{P}^t[\mathcal{I}]$, hence condition (iii) means that

$$\operatorname{centroid}_{\mathcal{Y}} f_k = \operatorname{centroid}_{\mathcal{X}} f_k, \tag{5.4}$$

for all $k \in [0,t]_{\mathbb{Z}}$.

Let $\mathbf{c} := (c_1,\dots,c_t)$ where $c_k := n \cdot \operatorname{centroid}_{\mathcal{X}} f_k$ for $k \in [1,t]_{\mathbb{Z}}$. Let $P_0 := P_{\mathbf{c}}$ be the positive integer defined in Remark 5.1. Then, both $\mathrm{J}_p(t;\mathcal{X};P)$ and $\widetilde{\mathrm{J}}_p(t;\mathcal{X};P)$ are empty when $P \notin P_0\,\mathbb{Z}_{>0}$. From now on, P is assumed to be in $P_0\,\mathbb{Z}_{>0}$

In this proof, we identify an integer-weighted levelling space $\mathcal{Y}$ with a tuple $\mathbf{y} = (y_1,\dots,y_n)$ with $y_1 \le \cdots \le y_n$, where y_i's are elements in $\mathcal{Y}$ and each $y \in \mathcal{Y}$ appears $\mu_Y(y)$ times in $\mathbf{y}$. Clearly, Eq. (5.4) holds for $k = 0$. For each $k \in [1,t]_{\mathbb{Z}}$, Eq. (5.4) holds for a levelling space $\mathcal{Y}$ satisfying conditions (i) and (ii), if and only if

$$\sum_{i=1}^{n} y_i^k = c_k,$$

since $\operatorname{centroid}_{\mathcal{Y}} f_k = \sum_{i=1}^n y_i^k/n$ and $\operatorname{centroid}_{\mathcal{X}} f_k = c_k/n$. Let $\mathrm{J}(t;\mathcal{X};P)$ be the set of levelling spaces $\mathcal{Y}$ satisfying conditions (i), (ii) and (iii). Then, there exists a bijection between $\mathrm{J}(t;\mathcal{X};P)$ and $\mathrm{J}(t,n;\mathbf{c};P)$ up to permutations of elements of a solution $(y_1,\dots,y_n)$.

Denote by $\mathrm{J}_p(t,n;\mathbf{c};P)$ the set of all solutions $(y_1,\dots,y_n)$ in $\mathrm{J}(t,n;\mathbf{c};P)$ such that all $p(y_i)$'s are distinct for $i \in [1,n]_{\mathbb{Z}}$, and let $\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)$ be the complement $\mathrm{J}(t,n;\mathbf{c};P) \setminus \mathrm{J}_p(t,n;\mathbf{c};P)$.

A levelling space $\mathcal{Y} \in \mathrm{J}(t;\mathcal{X};P)$ is in $\mathrm{J}_p(t;\mathcal{X};P)$ if and only if $p(y_i)$'s are distinct for $i \in [1,n]_{\mathbb{Z}}$, hence there exists a bijection between $\mathrm{J}_p(t;\mathcal{X};P)$ and $\mathrm{J}_p(t,n;\mathbf{c};P)$ up to permutations of elements of a solution. Therefore,

$$|\mathrm{J}_p(t;\mathcal{X};P)| \asymp_n |\mathrm{J}_p(t,n;\mathbf{c};P)|. \tag{5.5}$$

For each levelling space $\mathcal{Y} \in \widetilde{\mathrm{J}}_p(t;\mathcal{X};P)$, we have $y_i = y_j$ for some indexes i,j because μ_Y is not the counting measure, hence $\mathbf{y} \in \widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)$. Therefore,

$$|\widetilde{\mathrm{J}}_p(t;\mathcal{X};P)| \le |\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)|. \tag{5.6}$$

By Eq. (5.5) and Theorem 5.2(ii), we have

$$|\mathrm{J}_p(t;\mathcal{X};P)| \asymp_n |\mathrm{J}_p(t,n;\mathbf{c};P)| \le |\mathrm{J}(t,n;\mathbf{c};P)| \ll_n P^{n-t(t+1)/2},$$

which proves one direction of Eq. (5.2). On the other hand, for every solution $(y_1,\dots,y_n)$ of Eq. (5.1) with $p(y_u) = p(y_v)$ for some distinct $u,v \in [1,n]_{\mathbb{Z}}$, $(y_i :\ i \ne u,v)$ is a solution of the system of equations

$$\sum_{i=1}^{n-2} x_i^k = (\mathbf{c}_{y_u,y_v})_k, \quad k \in [1,t]_{\mathbb{Z}},$$

where $\mathbf{c}_{y_u,y_v} := (c_k - y_u^k - y_v^k :\ k \in [1,t]_{\mathbb{Z}})$. Let

$$(0,1)_{P^{-1}\mathbb{Z}} \times_p (0,1)_{P^{-1}\mathbb{Z}} := \{(r,s) \in (0,1)_{P^{-1}\mathbb{Z}} :\ p(r) = p(s)\}$$

be the fiber product. Then,

$$
\begin{aligned}
&|\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)| \\
&\leq \sum_{1\leq u<v\leq n} \sum_{\substack{x_u,x_v\in(0,1)_{P^{-1}\mathbb{Z}}\\ p(x_u)=p(x_v)}} |\mathrm{J}(t,n-2;\mathbf{c}_{x_u,x_v};P)| \\
&\ll_n |(0,1)_{P^{-1}\mathbb{Z}} \times_p (0,1)_{P^{-1}\mathbb{Z}}|\ P^{n-2-t(t+1)/2} && \text{Theorem 5.2(ii)} \\
&\ll_p P^{n-1-t(t+1)/2}. && p \text{ is nonconstant}
\end{aligned}
\tag{5.7}
$$

Since $\mathcal{X}$ is in $\mathrm{J}(t;\mathcal{X},P)$, the tuple $\mathbf{x}$ associated to $\mathcal{X}$ is in the set $\mathrm{J}(t,n;\mathbf{c};P)$. Therefore, by Theorem 5.2(vi) and Eqs. (5.5) and (5.7),

$$|\mathrm{J}_p(t;\mathcal{X};P)| \asymp_n |\mathrm{J}_p(t,n;\mathbf{c};P)| = |\mathrm{J}(t,n;\mathbf{c};P)| - |\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)| \gg_{t,n,p} P^{n-t(t+1)/2},$$

which proves the other direction of Eq. (5.2), and by Eqs. (5.6) and (5.7),

$$|\widetilde{\mathrm{J}}_p(t;\mathcal{X};P)| \leq |\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)| \ll_{t,\mathcal{X},p} P^{n-1-t(t+1)/2},$$

which proves Eq. (5.3). □

6. Algebraic path-connectivity

We focus in this section on the algebraic path-connectivity of a subset of $\mathbb{R}^d$. In § 6.1, we use approximation theory to prove Theorem 1.7, which is used in the proof of Theorem 7.1 to construct a good path in an open connected subset of $\mathbb{R}^d$. In § 6.2, we show in Proposition 6.3 that the real unit sphere is not $\mathbb{F}$-algebraically path-connected for any subfield $\mathbb{F} \subseteq \mathbb{R}$. In Proposition 6.6, we show that the real unit sphere does not satisfy a property weaker than algebraic path-connectivity either. Note that if this weaker property were obtained for the real unit sphere, we could obtain from it the existence of rational spherical designs. Propositions 6.3 and 6.6 are not used in the remainder of the article.

6.1. **Algebraic path-connectivity of open connected sets.** For a uniformly continuous map f between metric spaces, let ω_f be the corresponding modulus of continuity, namely

$$\omega_f(\delta) := \sup_{\substack{x,y\in \mathrm{domain}\, f\\ \mathrm{dist}(x,y)\leq\delta}} \mathrm{dist}(f(x),f(y)).$$

A function f is *Dini-Lipschitz continuous* provided that $\omega_f(\delta)\log\delta$ converges to 0 as $\delta\to 0$. For $n\in N$, the n-th *Chebyshev polynomial of the first kind*, denoted by T_n, is the unique polynomial satisfying $T_n(\cos x) = \cos nx$. The roots of T_n,

$$\boldsymbol{\alpha}_n := \{\alpha_{n,k} :\ k\in[1,n]_{\mathbb{Z}}\}, \text{ where } \alpha_{n,k} := \cos\frac{2k-1}{2n}\pi, \tag{6.1}$$

are called *Chebyshev nodes*. The Lagrange interpolation of a function f at a finite set $\mathbf{n}$ of distinct nodes, denoted by $L(f,\mathbf{n})$, is given explicitly by

$$L(f,\mathbf{n})(x) := \sum_{\alpha\in\mathbf{n}} \left(f(\alpha) \prod_{\beta\in\mathbf{n}\setminus\{\alpha\}} \frac{x-\beta}{\alpha-\beta} \right).$$

Clearly, if $\mathbf{n}\subseteq\mathbb{F}$ and $f(\mathbf{n})\subseteq\mathbb{F}$ for some field $\mathbb{F}$, then $L(f,\mathbf{n})\in\mathbb{F}[x]$. Proposition 6.1 is a classical result on Chebyshev interpolation $L(f,\boldsymbol{\alpha}_n)$ of a real-valued function f on the interval $[-1,1]$.

Proposition 6.1 ([Riv69, Eq. 4.1.11, Theorem 4.5]). *Let f be a uniformly continuous real-valued function on the interval $[-1,1]$, and let ω_f be the modulus of continuity of f. For every positive integer n,*

$$\|f - L(f, \boldsymbol{\alpha}_{n+1})\|_\infty < 6\left(\frac{2}{\pi}\log n + 5\right)\omega_f(1/n).$$

In particular, if f is Dini-Lipschitz continuous, then $L(f, \boldsymbol{\alpha}_n)$ converges to f uniformly as $n \to +\infty$. □

Corollary 6.2 is a high-dimensional version of Proposition 6.1.

Corollary 6.2. *Let $f : [-1,1] \to \mathbb{R}^d$ be a Dini-Lipschitz continuous function. Then $L(f, \boldsymbol{\alpha}_n)$ converges to f uniformly as $n \to +\infty$.*

Proof. Suppose that $f = (f_1, \dots, f_d)$. All f_i's are Dini-Lipschitz continuous. The result follows by applying Proposition 6.1 to each f_i. □

Proof of Theorem 1.7. In this proof, a collection of indexed objects is denoted by the same letter in bold. For instance, the collection of all y_i's is denoted by $\mathbf{y}$.

Let $X \subseteq Z \cap \mathbb{F}^d$ be a finite subset. It suffices to construct a polynomial path $p : [-1,1] \to \mathbb{R}^d$ given by polynomials with coefficients in $\mathbb{F}$, namely $p \in \mathbb{F}[x]^d$, such that $X \subseteq p((-1,1)_{\mathbb{F}}) \subseteq Z$.

Since Z is both open and connected in the Euclidean space $\mathbb{R}^d$, for some natural number n, there exists a piecewise linear path in Z with $(n+1)$-pieces and passing through all points in X at the boundary of pieces. Let $y_0 := -1$, $y_{n+1} := 1$ and $y_i := \alpha_{n,i}$ for $i \in [1,n]_{\mathbb{Z}}$ (see Eq. (6.1)). Then, there exists a piecewise linear map $\ell_{\mathbf{y}} : [-1,1] \to \mathbb{R}^d$ with $n+2$ nodes $\mathbf{y}$ satisfying $X \subseteq \{\ell_{\mathbf{y}}(y_i) : \ i \in [1,n]_{\mathbb{Z}}\}$. More precisely, $\ell_{\mathbf{y}}$ is a linear map on the interval $[y_i, y_{i+1}]$ for all $i \in [0,n]_{\mathbb{Z}}$.

Since $\operatorname{Im}\ell_{\mathbf{y}}$ is compact and Z is open, there exists an $\epsilon > 0$ such that for every $f : [-1,1] \to \mathbb{R}^d$ satisfying $\|f - \ell_{\mathbf{y}}\|_\infty < \epsilon$, we have $\operatorname{Im} f \subseteq Z$.

The piecewise linear map $\ell_{\mathbf{y}}$ is Dini-Lipschitz continuous, hence by Corollary 6.2, for some sufficiently large odd natural number $m \in \mathbb{N}$,

$$\|\ell_{\mathbf{y}} - L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn})\|_\infty < \epsilon/2.$$

Let $v_j \in \mathbb{F}$ be a to-be-determined number near $\alpha_{mn,j}$ for $j \in [1,mn]_{\mathbb{Z}}$, $u_i := v_{im-(m-1)/2} \in \mathbb{F}$ for $i \in [1,n]_{\mathbb{Z}}$, and we further set $u_0 := y_0 = -1 \in \mathbb{F}$ and $u_{n+1} := y_{n+1} = 1 \in \mathbb{F}$. Then, there exists a piecewise linear map $\ell_{\mathbf{u}}$ with $n+2$ nodes $\mathbf{u}$ such that $\ell_{\mathbf{u}}(u_i) = \ell_{\mathbf{y}}(y_i)$ for $i \in [0,n+1]_{\mathbb{Z}}$. It is clear that $\operatorname{Im}\ell_{\mathbf{u}} = \operatorname{Im}\ell_{\mathbf{y}}$. Now, for each $j \in [1,mn]_{\mathbb{Z}}$, let v_j be sufficiently close to $\alpha_{mn,j}$. For every $i \in [1,n]_{\mathbb{Z}}$, since $\alpha_{n,i} = \alpha_{mn,im-(m-1)/2}$, u_i is sufficiently close to $y_i = \alpha_{n,i}$ as well. Therefore, we choose a suitable $\mathbf{v}$ such that

$$\|L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn}) - L(\ell_{\mathbf{u}}, \mathbf{v})\|_\infty < \epsilon/2.$$

By subadditivity of norms,

$$\|\ell_{\mathbf{y}} - L(\ell_{\mathbf{u}}, \mathbf{v})\|_\infty \le \|\ell_{\mathbf{y}} - L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn})\|_\infty + \|L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn}) - L(\ell_{\mathbf{u}}, \mathbf{v})\|_\infty < \epsilon,$$

hence $\operatorname{Im} L(\ell_{\mathbf{u}}, \mathbf{v}) \subseteq Z$.

Note that, for $i \in [1,n]_{\mathbb{Z}}$, since $u_i = v_{im-(m+1)/2}$,

$$L(\ell_{\mathbf{u}}, \mathbf{v})(u_i) = L(\ell_{\mathbf{u}}, \mathbf{v})(v_{im-(m+1)/2}) = \ell_{\mathbf{u}}(v_{im-(m+1)/2}) = \ell_{\mathbf{u}}(u_i) = \ell_{\mathbf{y}}(y_i),$$

which implies that $X \subseteq \{L(\ell_{\mathbf{u}}, \mathbf{v})(u_i) : \ i \in [1,n]_{\mathbb{Z}}\} \subseteq L(\ell_{\mathbf{u}}, \mathbf{v})((-1,1)_{\mathbb{F}})$. Moreover, elements in $\mathbf{v}$ are in $\mathbb{F}$, points in X have coordinates in $\mathbb{F}$ and $\ell_{\mathbf{u}}$ is piecewise $\mathbb{F}$-linear, so we have $L(\ell_{\mathbf{u}}, \mathbf{v}) \in \mathbb{F}[x]^d$. Thus, $L(\ell_{\mathbf{u}}, \mathbf{v})$ is a desired path. □

6.2. **Non-algebraic path-connectivity of the real spheres.**

Proposition 6.3. *Let $d \in \mathbb{N}$. There are no non-constant polynomial paths on $\mathcal{S}^d$. In particular, the real sphere $\mathcal{S}^d$ is not $\mathbb{F}$-algebraically path-connected for any subfield $\mathbb{F} \subseteq \mathbb{R}$.*

Proof. Suppose that there exists a polynomial map $p = (p_0, \dots, p_d) \in \mathbb{R}[x]^d$ such that $p(I) \subseteq \mathcal{S}^d$ for some nontrivial interval I. Since the leading coefficient of each p_i^2 is positive and $\sum_{i=0}^d p_i^2 = 1$, every p_i is constant, which means that p is constant. $\square$

Let $\operatorname{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the algebra of integer-valued polynomials. Let Δ be the standard forward difference operator, namely $(\Delta f)(x) := f(x+1) - f(x)$ for a function f, and D the differential operator, namely $\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}\,x}$. Next, we present an elementary proof of a result on integer-valued polynomials. Note that this result can be essentially deduced from a stronger result in [DLS64], which is proved using algebraic number theory.

Theorem 6.4. *Let $f, g \in \operatorname{Int}(\mathbb{Z})$. If $f(\mathbb{N}) \subseteq g(\mathbb{N})$, then $f = g \circ h$ for some $h \in \operatorname{Int}(\mathbb{Z})$. Moreover, the polynomial h is unique unless either f or g is a constant.*

Proof. The result holds trivially when either f or g is a constant. From now on, we assume that f and g are not constants. The polynomials f and g are strictly monotone on a neighborhood of $+\infty$. So, for sufficiently large $x_0 \in \mathbb{N}$, the function $\eta := g^{-1} \circ f : \mathbb{R}_{\geq x_0} \to \mathbb{R}_{\geq 0}$ is well-defined, which has the order $\eta(x) \asymp x^{\frac{\deg f}{\deg g}}$ as $x \to +\infty$.

Let k be a natural number. We first compute the k-th derivative of η. For a rational function $r \in \mathbb{Q}(x)$ and a polynomial $p \in \mathbb{Q}[x]$,

$$\mathrm{D}((r \circ \eta) \cdot p) = \left(\frac{\mathrm{D}r}{\mathrm{D}g} \circ \eta\right) \cdot (\mathrm{D}f) \cdot p + (r \circ \eta) \cdot (\mathrm{D}p).$$

Analyzing the order of both sides the equation above, we have $\left(\mathrm{D}((r \circ \eta) \cdot p)\right)(x) \ll x^{-1}((r \circ \eta) \cdot p)(x)$ as $x \to +\infty$. By induction, it is easy to show that $(\mathrm{D}^k \eta)(x) = \left(\mathrm{D}^k((x \circ \eta) \cdot 1)\right) \ll x^{-k}\eta(x)$ as $x \to +\infty$. We now compute the k-th difference of η using the well-known relation

$$(\Delta^k \eta)(x) = (\mathrm{D}^k \eta)(x + \theta_x),$$

for some $\theta_x \in [0, k]$. For $k := \left\lceil \frac{\deg f}{\deg g} \right\rceil + 1$, we have

$$(\Delta^k \eta)(x) \asymp (\mathrm{D}^k \eta)(x) \ll x^{-k}\eta(x) \ll x^{-1},$$

as $x \to +\infty$. The condition $f(\mathbb{N}) \subseteq g(\mathbb{N})$ forces $\Delta^k \eta$ to take integer values on $\mathbb{Z}_{\geq x_0}$. So, there exists an $x_1 \in \mathbb{N}$ such that $(\Delta^k \eta)(n) = 0$ for all $n \in \mathbb{Z}_{\geq x_1}$, as a result, there exists a unique $h \in \operatorname{Int}(\mathbb{Z})$ such that η and h agree on $\mathbb{Z}_{\geq x_1}$. Thus, the polynomials $f = g \circ \eta$ and $g \circ h$ agree on $\mathbb{Z}_{\geq x_1}$, which implies that $f = g \circ h$. $\square$

Corollary 6.5. *Let $f : I \to \mathbb{R}$ be a continuous function on a real interval I such that $f(I \cap \mathbb{Q}) \subseteq \mathbb{Q}$ and f^2 is a polynomial function. Then, f is a piecewise polynomial function.*

Proof. The result holds trivially when f is a constant or I is trivial. Now, we only consider non-constant f and nontrivial I. By a linear change of variable with coefficients in $\mathbb{Q}$, without loss of generality, we assume that $[0, 1] \subseteq I$.

Let $q \in \mathbb{R}[x]$ be the polynomial such that $q = f^2$ on I. Since q maps rational numbers to rational numbers, $q \in \mathbb{Q}[x]$. Let c be a nonzero common multiple of the coefficients of the polynomial q, and let $r := c^2(x+1)^{2\deg q} q(\frac{1}{x+1}) \in \mathbb{Z}[x]$. Since $r = \left(c(x+1)^{\deg q} f(\frac{1}{x+1})\right)^2$ and $f(\frac{1}{x+1}) \in \mathbb{Q}$, we have $r(\mathbb{N}) \subseteq \{n^2 : n \in \mathbb{N}\}$. By Theorem 6.4, there exists a polynomial $h \in \mathbb{Q}[x]$ such that $h^2 = r$. So there exists a polynomial $p \in \mathbb{Q}[x]$ such that $q = p^2$, hence $|f| = |p|$ on I. Therefore, by continuity of f, f is a piecewise polynomial function. $\square$

Proposition 6.6. *There does not exist a non-constant continuous path $p = (p_0, \dots, p_d) : [0,1] \to \mathcal{S}^d$ such that for each i, p_i^2 is a polynomial function, and p maps rational points to rational points.*

Proof. Suppose that there exists a desired path p. By Corollary 6.5, each p_i is a piecewise polynomial function, so p is a piecewise polynomial path. Thus, on each piece of p, p is a constant path by Proposition 6.3. By the continuity of p, p is a constant path on $[0,1]$, which contradicts our assumption. ☐

7. Proofs of the main result and its corollaries

We first prove the main result in § 7.1, and then prove its corollaries in § 7.2. Note that we use Definition 3.1 as the definition of designs, unless stated explicitly otherwise.

7.1. Main result.

Theorem 7.1. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space satisfying Conditions 1.3 and 1.4, and let t be a natural number. Denote by $\mathrm{J}(t, n; \mathcal{Z}; P)$ the set of all rational t-designs $\mathcal{X}$ on $\mathcal{Z}$ of size n satisfying $\mathcal{X} \subseteq \mathcal{Z} \cap (P^{-1}\mathbb{Z})^d$. If $\mathcal{Z}$ is $\mathbb{Q}$-algebraically path-connected, then there exist positive integers d_0 and n_0 such that for every natural number $n \geq n_0$,*

$$|\mathrm{J}(t, n; \mathcal{Z}; P)| \gg_{t,n,\mathcal{Z}} P^{(n - td_0(td_0+1)/2)/d_0},$$

as $P \to \infty$ in the domain $P_0\mathbb{Z}_{>0}$ for some positive integer P_0.

The strategy for proving Theorem 7.1 and its corollary, Theorem 1.6, is as follows: In Step 1, we apply algebraic path-connectivity of $\mathcal{Z}$ to find a suitable path $\mathcal{I} \to \mathcal{Z}$. In Step 2, we use this path to construct a certain weighted levelling space in $\mathcal{I} \cap \mathbb{Q}$. In Step 3, we convert the measure of this levelling space to a measure taking integer values. In Step 4, we construct an equivalent integer-weighted levelling space in $\mathcal{I} \cap \mathbb{Q}$ for each sufficiently large total measure. In Step 5, we regard each integer-weighted levelling space as a multiset, and separate the repeated elements to obtain levelling spaces in $\mathcal{I} \cap \mathbb{Q}$ equipped with counting measures. In Step 6, we lift these levelling spaces to levelling spaces in $\mathcal{Z} \cap \mathbb{Q}^d$, which will be rational designs on $\mathcal{Z}$. At the end, in Step 7, we analyze the number of rational designs on $\mathcal{Z}$.

Proof. We are going to construct several levelling spaces $\mathcal{X}, \mathcal{X}', \mathcal{X}_n \subseteq \mathcal{Z} \cap \mathbb{Q}^d$ and $\mathcal{Y}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{Y}}_n, \mathcal{Y}_n \subseteq \mathcal{I} \cap \mathbb{Q}$, and $\mathcal{X}_n$ will be a rational t-design on $\mathcal{Z}$ of size n. In the following diagram, we summarize the sequence of constructions.

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\text{Lemma 3.4}"] & \mathcal{X} \arrow[r, "\text{Lemma 2.16}"] & \mathcal{X}' \arrow[d, bend left, "\text{Lemma 2.6(v)}"] & & & \mathcal{X}_n \\
\mathcal{I} \arrow[u, "p"] & & \mathcal{Y} \arrow[u, bend left, "p"] \arrow[r, "\text{Lemma 3.6}"'] & \widetilde{\mathcal{Y}} \arrow[r, "\text{Theorem 4.5}"'] & \widetilde{\mathcal{Y}}_n \arrow[r, "\text{Theorem 5.3}"'] & \mathcal{Y}_n \arrow[u, "p"]
\end{tikzcd}$$

Step 1. Since $\mathcal{Z}$ satisfies both Conditions 1.3 and 1.4, according to Lemma 3.4, there exists a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate (see Definition 2.12) weighted rational t-design $\mathcal{X}$ on $\mathcal{Z}$ with support X. By $\mathbb{Q}$-algebraic path-connectivity of $\mathcal{Z}$ (see Definition 1.5), there exists a polynomial path $p \in \mathbb{Q}[x]^d$ such that $X \subseteq p(\mathcal{I} \cap \mathbb{Q}) \subseteq \mathcal{Z} \cap \mathbb{Q}^d$. Let $t_0 := t \deg p$.

Step 2. Let S be a finite $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate subset of $\mathcal{I} \cap \mathbb{Q}$ such that $|S| \geq t_0$. Since X is the support of $\mathcal{X}$, which is a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design on $\mathcal{Z}$, according to Lemma 2.16, $X' := X \cup p(S)$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design $\mathcal{X}'$ on $\mathcal{Z}$. Let $Y := p^{-1}(X') \cap \mathcal{I} \cap \mathbb{Q}$. The map p then induces a surjection $p|_Y : Y \to X'$, so by Lemma 2.6(v), Y is the support of a levelling space $\mathcal{Y} \subseteq \mathcal{I} \cap \mathbb{Q}$ such that $p(\mathcal{Y}) = \mathcal{X}'$. Moreover, $S \subseteq Y$, hence $|Y| \geq |S| \geq t_0$ and $\mathcal{Y}$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate because S is $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate.

Step 3. Since $\mathcal{Y}$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate, by Lemma 3.6, there exists an integer-weighted (see Definition 2.2) levelling space $\widetilde{\mathcal{Y}} \subseteq \mathcal{I} \cap \mathbb{Q}$ whose support coincides with the support of $\mathcal{Y}$ and which is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent (see Definition 2.8) to $\mathcal{Y}$.
Step 4. By Theorem 4.5, there exists a natural number n_0, which we assume to be at least $3t_0^2 2_0^t - t_0$, such that for every $n \geq n_0$, there exists an integer-weighted levelling space $\widetilde{\mathcal{Y}}_n \subseteq \mathcal{I} \cap \mathbb{Q}$ whose total measure is n and which is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent to $\widetilde{\mathcal{Y}}$.
Step 5. Let $\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)$ be the set of levelling spaces $\mathcal{Y}_n \subseteq \mathcal{I} \cap Q^{-1}\mathbb{Z}$ such that $\mathcal{Y}_n$ has counting measure with total measure n, p is injective on $\widetilde{\mathcal{Y}}_n$, and $\mathcal{Y}_n$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent to $\widetilde{\mathcal{Y}}_n$. By Theorem 5.3,

$$|\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)| \asymp_{t_0, \widetilde{\mathcal{Y}}_n, p} Q^{n - t_0(t_0+1)/2},$$

as $Q \to \infty$ in the domain $Q_0 \mathbb{Z}_{>0}$ for some positive integer Q_0 depending on t_0 and $\widetilde{\mathcal{Y}}_n$.
Step 6. Let $P := Q^{\deg p}$. We will show that p induces an inclusion

$$\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q) \to \mathrm{J}(t, n; \mathcal{Z}; P).$$

Let $\mathcal{Y}_n$ be an arbitrary element in $\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)$ for some $Q \in Q_0 \mathbb{Z}_{>0}$. By the transitivity of $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalence, $\mathcal{Y}_n$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent to $\mathcal{Y}$. Let $\mathcal{X}_n := p(\mathcal{Y}_n) \subseteq \mathcal{Z} \cap (P^{-1}\mathbb{Z})^d$. By injectivity of p on $\mathcal{Y}_n$, $\mathcal{X}_n$ has the counting measure with total measure n. Since $\mathcal{Y}_n$ and $\mathcal{Y}$ are $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent and $p^* \mathcal{P}^t[\mathcal{Z}] \subseteq \mathcal{P}^{t_0}[\mathcal{I}]$, they are also $p^* \mathcal{P}^t[\mathcal{Z}]$-equivalent, where $p^* \mathcal{P}^t[\mathcal{Z}] = \{f \circ p : \ f \in \mathcal{P}^t[\mathcal{Z}]\}$ is the pullback of polynomials in $\mathcal{P}^t[\mathcal{Z}]$. Therefore, by Lemma 2.10, $\mathcal{X}_n = p(\mathcal{Y}_n)$ and $\mathcal{X}' = p(\mathcal{Y})$ are $\mathcal{P}^t[\mathcal{Z}]$-equivalent. Furthermore, since the levelling space $\mathcal{X}'$ is a weighted t-design on $\mathcal{Z}$, $\mathcal{X}_n$ is a rational t-design on $\mathcal{Z}$ of size n. So, p induces an inclusion that maps $\mathcal{Y}_n \in \mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)$ to $\mathcal{X}_n \in \mathrm{J}(t, n; \mathcal{Z}; P)$.
Step 7. Set $d_0 := \deg p$ and $P_0 := Q_0^{d_0}$. Using the inclusion obtained from Step 6 and then asymptotic formula in Step 5,

$$|\mathrm{J}(t, n; \mathcal{Z}; P)| \geq |\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)| \asymp_{t_0, \widetilde{\mathcal{Y}}_n, p} Q^{n - t_0(t_0+1)/2} = P^{(n - td_0(td_0+1)/2)/d_0},$$

as $P \to \infty$ in the domain $P_0 \mathbb{Z}_{>0}$, where $d_0 := \deg p$, t_0 and p are determined by $t, n, \mathcal{Z}$ in Step 1, and $\widetilde{\mathcal{Y}}_n$ is determined by $t, n, \mathcal{Z}$ in Step 4. ☐

Proof of Theorem 1.6. By Theorem 7.1, there exists an n_0 such that for all natural number $n \geq n_0$, the set $\mathrm{J}(t, n; \mathcal{Z}; P)$ is nonempty for some P, hence there exists a rational t-design $\mathcal{X}$ on $\mathcal{Z}$ of size n in the sense of Definition 3.1. According to Remark 3.2, the support of $\mathcal{X}$ is a rational t-design on $\mathcal{Z}$ of size n in the sense of Definition 1.1. ☐

7.2. **Corollaries of the main result.** Combining Theorem 1.6 and Theorem 1.7, we get an immediate corollary as follows.

Theorem 7.2. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ satisfy Conditions 1.3 and 1.4, and let t be a natural number. The following statements hold.*

(i) *If $\mathcal{Z}$ is open and connected, then there exist rational t-designs on $\mathcal{Z}$ of all sufficiently large sizes.*
(ii) *If $\mathcal{Z}$ is a d-dimensional convex set equipped with the Lebesgue measure, then there exist rational t-designs on $\mathcal{Z}$ of all sufficiently large sizes.*

Proof. (i) Since $\mathcal{Z}$ is open, Condition 1.3 holds. According to Theorem 1.7, since we further have that $\mathcal{Z}$ is connected, it is consequently $\mathbb{Q}$-algebraically path-connected. The result then follows from Theorem 1.6.
(ii) Let $\operatorname{int} \mathcal{Z}$ be the interior of $\mathcal{Z}$. Since $\dim \mathcal{Z} = d$, $\operatorname{int} \mathcal{Z}$ is nonempty. By the convexity of $\mathcal{Z}$, $\operatorname{int} \mathcal{Z}$ is open and convex, hence open and connected. By (i), there exist rational t-designs on $\operatorname{int} \mathcal{Z}$. Since the boundary of the convex set $\mathcal{Z}$ has Lebesgue measure 0, a rational t-design on $\operatorname{int} \mathcal{Z}$ is automatically a rational t-design on $\mathcal{Z}$. ☐

Theorem 1.8 and Theorem 1.9 are corollaries of Theorem 7.2.

Proof of Theorem 1.8. It is easy to check that d-dimensional convex polytopes given by inequalities with rational coefficients satisfy Conditions 1.3 and 1.4. The result follows from Theorem 7.2(ii) applied to $\mathcal{Z}$. $\square$

Proof of Theorem 1.9. Consider the $(d-1)$-sphere $\mathcal{S}^{d-1}$ in $\mathcal{S}^d$ given by $\{(x_0,\dots,x_d)\in\mathcal{S}^d:\ x_0=0\}$. The $(d-1)$-sphere is closed and has measure 0. So, we get a levelling space $\mathcal{S}^d\setminus\mathcal{S}^{d-1}$.

Let $p:\ \mathcal{S}^d\to\mathbb{R}^d$ be the projection map given by $(x_0,\dots,x_d)\mapsto(x_1,\dots,x_d)$, and let $\mathcal{B}^d$ be the image of $\mathcal{S}^d\setminus\mathcal{S}^{d-1}$ under p. As a topological space, $\mathcal{B}^d$ is a d-dimensional real unit open ball, and it is equipped with the pushforward measure (see Definition 2.5). By Theorem 7.2(i), there exists a rational t-design $\mathcal{X}$ on $\mathcal{B}^d$. Let $\mathcal{Y}:=p^{-1}(\mathcal{X})$ be defined as in Lemma 2.6(v). For each point in $\mathcal{Y}$, all its coordinates are rational except possibly the first coordinate. By Lemma 2.6(vi), $2\,\mathcal{Y}$ is equipped with the counting measure. We claim that $2\,\mathcal{Y}$ is a spherical t-design on $\mathcal{S}^d$, hence the result.

It suffices to show that for every monic monomial $f\in\mathbb{R}[\mathcal{S}^d]$, Eq. (1.1) holds for $2\,\mathcal{Y}$. Indeed, the vector space $\mathbb{R}[\mathcal{S}^d]$ has a decomposition $\mathbb{R}[\mathcal{B}^d]\oplus x_0\,\mathbb{R}[\mathcal{B}^d]$. For each $f\in x_0\,\mathbb{R}[\mathcal{B}^d]$, it is easy to see by symmetry that both sides of Eq. (1.1) are 0, and for each $f\in\mathbb{R}[\mathcal{B}^d]$, Eq. (1.1) follows from the fact that $\mathcal{X}$ is a t-design on $\mathcal{B}^d$. Therefore, $2\,\mathcal{Y}$ is a spherical t-design. $\square$

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