RATIONAL DESIGNS

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Abstract. The existence of designs over $\mathbb{R}$ on a path-connected space has been proved by Seymour and Zaslavsky. This paper studies for the first time rational designs, that is designs over $\mathbb{Q}$. We establish the existence of rational designs on an open connected space under certain necessary conditions. Consequently, we show that there exist rational designs on rational simplicial complexes and spherical designs over some real abelian extension of $\mathbb{Q}$.

Contents

1. Introduction

Designs are finite subsets of a given space that approximate the whole space nicely. In this paper, we focus on geometric designs, which are also known as cubature formula [Möl79; GS81] and averaging sets [SZ84; Wag91]. They were first introduced by Delsarte, Goethals and Seidel in [DGS77] for spheres, and later studied for other spaces.

1.1. Rational spherical designs.

Definition 1.1 ([DGS77])**.** Let $\mathcal{S}^d$ be the d-dimensional real unit sphere equipped with the spherical measure ν^d. A *spherical design of strength t*, or simply a *spherical t-design*, on $\mathcal{S}^d$ is a nonempty finite set $X \subseteq \mathcal{S}^d$ such that

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{\nu^d(\mathcal{S}^d)}\int_{\mathcal{S}^d} f \,\mathrm{d}\,\nu^d, \tag{1.1}$$

for every polynomial $f \in \mathbb{R}[x_0, \dots, x_d]$ of degree at most t. When $X \subseteq \mathbb{F}^{d+1}$ for some subfield $\mathbb{F}$ of $\mathbb{R}$, we call X a spherical *t-design over* $\mathbb{F}$. A *rational spherical t-design* is a spherical t-design over $\mathbb{Q}$.

The existence of spherical t-designs over $\mathbb{R}$ for all natural numbers t was long established since Seymour and Zaslavsky [SZ84]. However, the existence of spherical t-designs over a proper subfield $\mathbb{F} \subseteq \mathbb{R}$ is only known for special cases. Kuperberg [Kup05] constructed spherical t-designs on $\mathcal{S}^2$ over $\mathbb{Q}^{\mathrm{alg}} \cap \mathbb{R}$, where $\mathbb{Q}^{\mathrm{alg}}$ is the algebraic closure of $\mathbb{Q}$. Inspired by the ideas of this paper, Xiang [Xia18] constructed spherical t-designs on $\mathcal{S}^d$ over $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$, where $\mathbb{Q}^{\mathrm{ab}}$ is the abelian closure of $\mathbb{Q}$. The following two famous examples over $\mathbb{Q}$ have been discovered by Venkov [Ven84].

Example 1.2. For an integral lattice Λ and a natural number m, let $\Lambda_m := \{x \in \Lambda : \|x\|^2 = m\}$ be the shell of lattice points of norm m.

(i) Let $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^8$ be the E_8-lattice. For every $m \in \mathbb{Z}_{\geq 1}$, $\frac{1}{2m}\Lambda_{4m^2}$ is a rational spherical 7-design on $\mathcal{S}^7$.

(ii) Let $\Lambda \subseteq \frac{1}{\sqrt{8}}\mathbb{Z}^{24}$ be the Leech lattice. For every $m \in \mathbb{Z}_{\geq 2}$, $\frac{1}{\sqrt{2}m}\Lambda_{2m^2}$ is a rational spherical 11-design on $\mathcal{S}^{23}$.

The current known methods cannot construct rational spherical designs of high strengths [BB09]. Shells of lattices, such as the ones in Example 1.2, are only known to be spherical designs of strength at most 11, and Example 1.2(ii) yields a spherical 12-design if and only if some generalization of Lehmer's conjecture does not hold. On $\mathcal{S}^d$ with $d \geq 2$, orbits of a point under the action of finite reflection groups can only yield spherical designs of strength at most 19[1], and they are rarely rational.

Most proofs on the existence of spherical designs rely on the completeness of the field $\mathbb{R}$. As $\mathbb{Q}$ is not complete in $\mathbb{R}$, they cannot be adapted to prove the existence of rational spherical designs. Perturbing real spherical designs does not work either, since it will inevitably introduce weights of points in the designs, which are not allowed in the definition of spherical designs. In fact, the existence of rational spherical 12-designs is still unknown.

It is then natural to ask: over what small fields, there exist spherical designs on $\mathcal{S}^d$ for all strengths. Theorem 1.3 proves the existence of spherical designs for all strengths over some real abelian extension of $\mathbb{Q}$, which also provides a new proof of the existence of spherical designs over $\mathbb{R}$. The proof of Theorem 1.3, presented in § 7.2, employs results from analytic number theory, approximation theory and convex geometry, which let us overcome major difficulties arising from the incompleteness of proper subfields of $\mathbb{R}$.

[1]For types A-F, one can apply [Ban84, Theorem 1] to the corresponding root system, whose strength can be found in [BB09], from which it follows that a transitive orbit is of strength at most 15. For types H_3 and H_4, [Neu83] and [NSJ85] proved that a transitive orbit is of strength at most 10 and 19, respectively.

Theorem 1.3. *Let t be a natural number, and let d be a positive integer. There exists a natural number n_0 such that for every even natural number $n \geq n_0$, there exist spherical t-designs on $\mathcal{S}^d$ of size n over the field $\mathbb{Q}(\{\sqrt{q} : q \text{ prime}\})$. Furthermore, the points in the design can be chosen in a way that all their coordinates are rational numbers except possibly for the first coordinate.*

It is well-known that the sizes of spherical t-designs admit the Fisher type lower bound [DGS77], and this lower bound can be achieved asymptotically [BRV13]. Given a subfield $\mathbb{F} \subseteq \mathbb{R}$, it is an interesting problem to determine the smallest spherical t-designs over $\mathbb{F}$. In theory, the n_0 in Theorem 1.3 can be computed effectively, which gives an upper bound on the smallest size.

1.2. **Rational geometric designs.** In this section, we consider designs on general spaces $\mathcal{Z}$. We call a topological space equipped with a good measure a *levelling space* (see Definition 2.1), and define *(unweighted) designs* and *weighted designs* on levelling spaces in Definition 3.1. This definition was essentially given by Seymour and Zaslavsky [SZ84], who also proved that designs on $\mathcal{Z}$ over $\mathbb{R}$ always exist if the space $\mathcal{Z}$ is path-connected. There is a large body of literature on designs on general spaces, such as: complex spherical designs [RS14], Euclidean designs [NS88; DS89], designs on projective spaces [Hog82], designs on Grassmannians [BCN02; BBC04], designs on (projective) unitary groups [Sco08; RS09], and designs on polynomial spaces [God88; BBD99].

In many talks, Bannai proposed the question of the existence of rational t-designs on a given space $\mathcal{Z}$ for all natural number t. The similar question for weighted rational designs has a simple answer: we prove in § 3 that Condition 3.2 on the space $\mathcal{Z}$ is a necessary and sufficient condition for the existence of weighted rational designs. However, (unweighted) rational designs are much more difficult to find, and we aim to provide first information on their existence. In particular, we prove the following main result on open connected spaces.

Theorem 1.4. *Let $\mathcal{Z}$ be a levelling space such that $\mathcal{Z}$ is a topological subspace of $\mathbb{R}^d$ with $d \geq 1$, and assume that $\mathcal{Z}$ satisfies Condition 3.2. Let t be a natural number. If $\mathcal{Z}$ is open connected, then there exists a natural number n_0 such that for every natural number $n \geq n_0$, there exist rational t-designs on $\mathcal{Z}$ of size n.*

Theorem 1.4 follows from the stronger Theorem 7.1, where we prove the existence of rational designs on algebraically path-connected spaces (see § 6). Theorems 1.4 and 7.1 are proved in § 7.2 and § 7.1, respectively.

As a corollary of Theorem 1.4, we show the existence of rational designs on rational simplicial complexes in Theorem 1.5, which is proved in § 7.3.

Theorem 1.5. *Let $\mathcal{Z}$ be a simplicial k-complex in $\mathbb{R}^d$ for some positive integer k, equipped with the k-dimensional Hausdorff measure. Suppose that* *the vertices of $\mathcal{Z}$ are rational points. Then, there exist rational t-designs on $\mathcal{Z}$ for all natural numbers t.*

Another corollary of Theorem 1.4 is the existence of rational interval t-designs. We provide a more direct proof in Corollary 5.4.

1.3. **Some open problems.** It is still unknown whether rational spherical t-designs on $\mathcal{S}^d$ exist. The smallest open case is the existence of rational 4-designs on the unit circle $\mathcal{S}^1$. It is roughly equivalent to the existence of distinct rationals $x_1, \ldots, x_n$ such that $(1 - x_i^2)^{1/2} \in \mathbb{Q}$ and

$$\frac{1}{n} \sum_{i=1}^{n} x_i^4 = \frac{3}{8}.$$

If we parametrize rational points on $\mathcal{S}^1$, then we would get a Diophantine equation whose degree is larger than the number of variables, and the authors could not apply classical circle method in analytic number theory to solve it.

Under the necessary conditions, path-connectedness and algebraically path-connectedness of the space $\mathcal{Z}$ guarantee the existence of designs on $\mathcal{Z}$ over reals and over rationals, respectively. There are many spaces $\mathcal{Z}$ which are not path-connected, yet we can still find designs on $\mathcal{Z}$ over reals. Euclidean designs are such examples. It is an interesting problem to find a sufficient condition of some other type for the existence of designs, which may shed some light on the existence of rational spherical designs.

One referee asks if it is possible to find spherical t-designs $\mathcal{X}$ on $\mathcal{S}^d$ such that the inner products between points in $\mathcal{X}$ are all rational. It turns out that finding such spherical designs is tantamount to finding rational designs on some ellipsoid. Similar to rational designs on spheres, Theorem 1.4 cannot be applied to find rational designs on ellipsoids.

1.4. **Structure of the paper.** The paper is organized as follows. In § 2, we introduce the concept of *levelling space*, which provides a framework to talk about designs in a general setup. We then discuss properties of designs and construct weighted designs using convex geometry. In § 3, we prove that Condition 3.2 is a necessary and sufficient condition of the existence of weighted rational designs. In § 4, we analyze the possible total measures of integer-weighted designs. § 5 provides the analytic number theoretic arguments which we need in § 7. § 6 is concerned with algebraic path-connectivity and proves that open connected spaces are algebraically path-connected spaces using approximation theory. Finally, in § 7, we prove Theorem 7.1, from which the main result Theorem 1.4 follows, together with its corollaries, including Theorems 1.3 and 1.5.

Notation. For a subset $S \subseteq \mathbb{R}$ and a real interval $I \subseteq \mathbb{R}$, let $I_S := I \cap S$. For instance, for $a, b \in \mathbb{R}$, $(a, b)_{\mathbb{Q}}$ consists of all rational numbers in (a, b). Another example is $[0, t]_{\mathbb{Z}}$, which consists of all natural numbers no greater than t. The d-dimensional real unit sphere equipped with the spherical measure is denoted by $\mathcal{S}^d$, and the open unit interval $(0,1)$ equipped with the Lebesgue measure is denoted by $\mathcal{I}$. The number of points in a space $\mathcal{X}$ (or X) is denoted by $\#\mathcal{X}$ (or $\#X$). The total measure of a measure space $\mathcal{X}$ is denoted by $|\mathcal{X}|$.

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2. Levelling spaces and designs

In this section, we discuss certain topological measure spaces which we call *levelling spaces*. Some basic properties and constructions of levelling spaces are given in § 2.1. In § 2.2, we first define an equivalence relation of levelling spaces, and then define designs on a levelling space. In § 2.3, we apply some results from convex geometry to construct weighted designs on arbitrary levelling spaces.

2.1. Levelling spaces and their properties.

A *strictly positive measure space* $\mathcal{X} = (X, \mu_X)$ is a Hausdorff topological space X equipped with a measure μ_X such that every nonempty open set has positive measure. In particular, for a discrete space this means that all points have positive measures. The set X is said to be the *support* of $\mathcal{X}$.

Definition 2.1. A *levelling space* $\mathcal{X}$ is a nonempty strictly positive measure space of finite total measure $|\mathcal{X}|$. It is called *finite* if its *size* $\#\mathcal{X}$ is finite.

Throughout this paper, a levelling space $\mathcal{X}$ is always written in calligraphy font $\mathcal{X}$ in order to distinguish the levelling space $\mathcal{X}$ from its support, the topological space X. For two levelling spaces $\mathcal{X}$ and $\mathcal{Z}$, by saying $\mathcal{X}$ *is a subspace of* $\mathcal{Z}$, denoted by $\mathcal{X} \subseteq \mathcal{Z}$, we only mean that the support X of $\mathcal{X}$ is a topological subspace of the support Z of $\mathcal{Z}$. In particular, we do not assume any relation between the measures μ_X and μ_Z. Similarly, we may write $\mathcal{X} \subseteq Z$ or $X \subseteq \mathcal{Z}$ to mean that X is a topological subspace of Z.

Definition 2.2. Let $k \subseteq \mathbb{R}$ be a subset. A levelling space $\mathcal{X} = (X, \mu_X)$ is *k-weighted* if the image of μ_X is in k.

Typically, we take k to be either $\mathbb{Z}$ or $\mathbb{Q}$, and call the levelling space *integer-weighted* or *rational-weighted*, respectively. Recall that, as a measure space, the image of μ_X is always nonnegative. A finite levelling space $\mathcal{X}$ is just a nonempty finite weighted set (X, ω_X) where $\omega_X : X \to \mathbb{R}_{>0}$.

In particular, a finite integer-valued levelling space may be viewed as a nonempty finite multi-set.

Now, we introduce some operations on levelling spaces.

Definition 2.3. Let $\mathcal{X} = (X, \mu_X)$, and $\mathcal{Y} = (Y, \mu_Y)$ be two levelling spaces whose subtopologies on the set intersection $X \cap Y$ agree. The *sum of* $\mathcal{X}$ *and* $\mathcal{Y}$, denoted by $\mathcal{X} + \mathcal{Y}$, is defined to be $(X \cup Y, \mu_X + \mu_Y)$, where $X \cup Y$ is the topological union of X and Y, and $\mu_X + \mu_Y$ is the sum of the measures μ_X and μ_Y. More precisely, a subset $E \subseteq X \cup Y$ is measurable in $\mathcal{X} + \mathcal{Y}$ if and only if $E \cap X$ and $E \cap Y$ are measurable in $\mathcal{X}$ and $\mathcal{Y}$, respectively, and the measure is defined via $(\mu_X + \mu_Y)(E) := \mu_X(E \cap X) + \mu_Y(E \cap Y)$.

Definition 2.4. Let $\mathcal{X} = (X, \mu_X)$ be a levelling space, and let c be a positive real number. We set $c\,\mathcal{X} := (X, c\ \mu_X)$, where $c\ \mu_X$ is a constant scalar of μ_X.

Definition 2.5. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. For a levelling space $\mathcal{W} = (W, \mu_W)$ with $W \subseteq X$, the *image of* $\mathcal{W}$ *under* f is defined to be $f(\mathcal{W}) := (f(W), f_*\mu_W)$, where $f(W)$ is the image of W under f as a topological space, and $f_*\mu_W$ is the pushforward of the measure μ_W. That is to say, a subset $E \subseteq f(W)$ is measurable in $f(\mathcal{W})$ if and only if $f^{-1}(E) \cap W$ is measurable in $\mathcal{W}$, and $(f_*\mu_W)(E) := \mu_W(f^{-1}(E) \cap W)$.

Lemma 2.6 shows that the sums, positive scalars, and images of levelling spaces are still levelling spaces. It also provides some other constructions which we need later.

Lemma 2.6. *Let X and Y be two topological spaces. The following statements hold.*

(i) *Suppose that X and Y agree on $X \cap Y$. For levelling spaces $\mathcal{X}$ with support X and $\mathcal{Y}$ with support Y, their sum $\mathcal{X} + \mathcal{Y}$ is a levelling space.*
(ii) *Let $\mathcal{X}$ be a levelling space, and let c be a positive real number. Then, $c\,\mathcal{X}$ is a levelling space.*
(iii) *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be levelling spaces whose supports agree on pairwise intersections, and let $c_1, \dots, c_n$ be nonnegative real numbers that are not all zeros. Then, the linear combination $\sum_{i=1}^n c_i\,\mathcal{X}_i$ is a levelling space.*
(iv) *Let $f : X \to Y$ be a continuous map. For a levelling space $\mathcal{W} \subseteq X$, the image $f(\mathcal{W})$ is a levelling space.*
(v) *Let $f : X \to Y$ be a continuous map such that each of its fibers is finite. For a finite levelling space $\mathcal{W}$ with support $W \subseteq f(X)$, there exists a levelling space $f^{-1}(\mathcal{W})$ with support $f^{-1}(W)$ such that $f(f^{-1}(\mathcal{W})) = \mathcal{W}$ and all points in each fiber have the same measure.*
(vi) *Let $f : X \to Y$ be an n-fold covering. For a finite levelling space $\mathcal{W} \subseteq Y$ equipped with the counting measure, the levelling space $nf^{-1}(\mathcal{W})$ is equipped with the counting measure.*

Proof. (i): Let E be an open set in $X \cup Y$. The restriction $E \cap X$ and $E \cap Y$ are open in X and Y, respectively. Then, we have $(\mu_X + \mu_Y)(E) = \mu_X(E\cap X)+\mu_Y(E\cap Y) > 0$. Moreover, the total measure of $\mathcal{X}+\mathcal{Y}$ is the sum of total measures of X and Y, hence finite. Therefore, $\mathcal{X}+\mathcal{Y}$ is a levelling space.
(ii): A positive scalar of a strictly positive measure with finite total measure is still a strictly positive measure with finite total measure, and the result follows.
(iii): Corollary of (i) and (ii).
(iv): Let E be an open set in $f(W)$. Since f is continuous, the set $f^{-1}(E)\cap W$ is open in W. Then, $(f_*\mu_W)(E) = \mu_W(f^{-1}(E)\cap W) > 0$. The total measure of $f(\mathcal{W})$ is the same as the total measure of $\mathcal{W}$, hence finite. Therefore, $f(\mathcal{W})$ is a levelling space.
(v) Let $f^{-1}(\mathcal{W})$ be the levelling space with the support $f^{-1}(W)$ equipped with the measure μ such that $\mu(x) := \frac{\mu_W(f(x))}{\# f^{-1}(f(x))}$ for every point $x \in f^{-1}(W)$, which is well-defined since both W and $f^{-1}(W)$ are finite. It is then clear that $f(f^{-1}(\mathcal{W})) = \mathcal{W}$.
(vi): The result follows from the construction of $f^{-1}(\mathcal{W})$ in (v). □

2.2. **Equivalent levelling spaces and designs.** We say that a function f is *integrable* on a levelling space $\mathcal{X} = (X, \mu_X)$ if $f|_X$ is μ_X-integrable on X. In this section, we fix a levelling space $\mathcal{Z}$ and a finite dimensional real vector space V of continuous integrable real-valued functions on $\mathcal{Z}$.

Definition 2.7. Let $\mathcal{X} \subseteq \mathcal{Z}$ be a levelling space such that all functions in V are integrable on $\mathcal{X}$. For each $f \in V$, the *centroid of f on $\mathcal{X}$* is

$$\operatorname{centroid}_{\mathcal{X}} f := \frac{1}{|\,\mathcal{X}\,|}\int_X f \,\mathrm{d}\,\mu_X.$$

The *centroid of V on $\mathcal{X}$*, denoted by $\operatorname{centroid}_{\mathcal{X}} V$, is defined to be an element in the *dual space* V^*:

$$\begin{array}{rccc} \operatorname{centroid}_{\mathcal{X}} V : & V & \to & \mathbb{R} \\ & f & \mapsto & \operatorname{centroid}_{\mathcal{X}} f. \end{array}$$

Definition 2.8. Two levelling spaces $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{Z}$ are *V-equivalent* provided that $\operatorname{centroid}_{\mathcal{X}} V = \operatorname{centroid}_{\mathcal{Y}} V$.

Clearly, V-equivalence is an equivalence relation of levelling spaces that are contained in $\mathcal{Z}$.

Lemma 2.9. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n \subseteq \mathcal{Z}$ be levelling spaces that are V-equivalent to a levelling space $\mathcal{X} \subseteq \mathcal{Z}$, and let $c_1, \ldots, c_n$ be nonnegative real numbers that are not all zeros. Then, the linear combination $\sum_{i=1}^n c_i\,\mathcal{X}_i$ is also V-equivalent to $\mathcal{X}$.*

Proof. By Lemma 2.6(iii), the linear combination $\mathcal{Y} := \sum_{i=1}^{n} c_i \mathcal{X}_i$ is a levelling space. For every $f \in V$,

$$\operatorname{centroid}_{\mathcal{Y}} f = \frac{1}{|\mathcal{Y}|} \int_Y f \,\mathrm{d}\,\mu_Y = \frac{\sum_{i=1}^{n} c_i \int_{X_i} f \,\mathrm{d}\,\mu_{X_i}}{\sum_{i=1}^{n} c_i |\mathcal{X}_i|}.$$

For each i, $\frac{\int_{X_i} f \,\mathrm{d}\,\mu_{X_i}}{|\mathcal{X}_i|} = \operatorname{centroid}_{\mathcal{X}_i} f = \operatorname{centroid}_{\mathcal{X}} f$ by V-equivalence between $\mathcal{X}_i$ and $\mathcal{X}$. Therefore, $\operatorname{centroid}_{\mathcal{Y}} f = \operatorname{centroid}_{\mathcal{X}} f$, hence the result. □

Lemma 2.10. *Let $p\colon\ \mathcal{Z}' \to \mathcal{Z}$ be a continuous map between levelling spaces $\mathcal{Z}'$ and $\mathcal{Z}$, and let $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{Z}'$ be levelling spaces. Then, $p(\mathcal{X}_1)$ and $p(\mathcal{X}_2)$ are V-equivalent levelling spaces in $\mathcal{Z}$ if and only if $\mathcal{X}_1$ and $\mathcal{X}_2$ are p^*V-equivalent levelling spaces in $\mathcal{Z}'$, where $p^*V := \{f \circ p\colon\ f \in V\}$.*

Proof. It is easy to check that $\operatorname{centroid}_{p(\mathcal{X})} f = \operatorname{centroid}_{\mathcal{X}} f \circ p$ for every $f \in V$ and $\mathcal{X} \in \{\mathcal{X}_1, \mathcal{X}_2\}$, hence the result. □

Definition 2.11. A levelling space $\mathcal{X} \subseteq \mathcal{Z}$ is a *weighted V-design on $\mathcal{Z}$* if $\mathcal{X}$ is V-equivalent to $\mathcal{Z}$. We say that a weighted V-design on $\mathcal{Z}$ is a *V-design on $\mathcal{Z}$* if the associated measure μ_X is the counting measure, namely $\mu_X(x) = 1$ for every $x \in \mathcal{X}$.

The levelling space $\mathcal{Z}$ itself is clearly a weighted V-design on $\mathcal{Z}$.

2.3. **Existence of weighted designs.** Recall that we have fixed a levelling space $\mathcal{Z}$ and a finite dimensional real vector space V of continuous integrable real-valued functions on $\mathcal{Z}$. We equip V with the finest locally convex topology on V, and view V as a real topological vector space. The dual space V^* is then also a real topological vector space with $\dim V^* = \dim V$. If we choose a basis for V and its corresponding dual basis for V^*, then we can identify V and V^* with $\mathbb{R}^{\dim V}$. Moreover, the finest locally convex topologies on V and V^* are homeomorphic to the natural topology on $\mathbb{R}^{\dim V}$. The topological structure of V^* allows us to talk about limit, interior, and dimension of some subsets of V^*.

Definition 2.12. A subset $X \subseteq \mathcal{Z}$ is called *V-nondegenerate* provided that the restriction $V|_X := \{f|_X\colon\ f \in V\}$ satisfies $\dim V|_X = \dim V$.

In other words, X is V-nondegenerate if and only if the natural epimorphism $V \twoheadrightarrow V|_X$ is an isomorphism.

Let S be a subset of the real topological vector space V^*. The *convex cone of S*, *convex hull of S* and the *affine space generated by S* are the sets of all nonnegative, convex and affine linear combinations of S. They are denoted by $\operatorname{cone} S$, $\operatorname{conv} S$ and $\operatorname{aff} S$, respectively. Denote $\operatorname{int} S$ the *interior of S in V^**, and denote $\operatorname{reint} S$ the *relative interior of S*, that is, the interior of S in $\operatorname{aff} S$.

For a subset $X \subseteq \mathcal{Z}$, we define a subset $\operatorname{ev}_X V$ of V^* as

$$\operatorname{ev}_X V := \{\operatorname{ev}_x V \in V^*\colon\ x \in X\},$$

where $(\operatorname{ev}_x V)(f) := f(x)$ for all $f \in V$.

Definition 2.13. Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield and V a real vector space. An $\mathbb{F}$-*structure on* V is an $\mathbb{F}$-vector space W such that $V \cong \mathbb{R} \otimes_{\mathbb{F}} W$.

An $\mathbb{F}$-structure W on V gives an $\mathbb{F}$-structure $W^* := \operatorname{Hom}_{\mathbb{F}}(W, \mathbb{F})$ on V^* since $V^* \cong \mathbb{R} \otimes_{\mathbb{F}} W^*$. This isomorphism enables us to view W^* as an $\mathbb{F}$-vector subspace of V^*. A typical example of $\mathbb{F}$-structure comes from a choice of basis of the vector space. Let $\{f_i\}$ be an $\mathbb{R}$-basis of a finite dimensional real vector space V. Then, the $\mathbb{F}$-vector space $W := \bigoplus_i \mathbb{F} f_i$ is an $\mathbb{F}$-structure of V. Moreover, let $\{f_i^*\}$ be the dual basis on V^*, then the $\mathbb{F}$-vector space $W^* = \bigoplus_i \mathbb{F} f_i^*$ is the corresponding $\mathbb{F}$-structure on V^*.

Next, we explore some properties of the centroid of V on $\mathcal{X}$ and the convex cone of $\operatorname{ev}_X V$.

Proposition 2.14. *Let $X \subseteq \mathcal{Z}$ be a subset. Assume that the constant function $1_{\mathcal{Z}}$ on $\mathcal{Z}$ is in the vector space V. The following statements hold.*

- (i) *Let $A := \{\alpha \in V^* : \alpha(1_{\mathcal{Z}}) = 1\}$ be an affine subspace of V^*. Then we have* $(\operatorname{cone} \operatorname{ev}_X V) \cap A = \operatorname{conv} \operatorname{ev}_X V$.
- (ii) *The set X is V-nondegenerate if and only if* $\dim \operatorname{cone} \operatorname{ev}_X V = \dim V$. *In particular,* $\operatorname{int} \operatorname{cone} \operatorname{ev}_X V = \emptyset$ *if X is not V-nondegenerate, and* $\operatorname{int} \operatorname{cone} \operatorname{ev}_X V = \operatorname{reint} \operatorname{cone} \operatorname{ev}_X V \neq \emptyset$ *if X is V-nondegenerate.*
- (iii) *If X is the support of a V-nondegenerate weighted V-design $\mathcal{X}$ on $\mathcal{Z}$, then* $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$.
- (iv) *If* $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$, *then* $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_Y V$ *for some finite subset $Y \subseteq X$.*
- (v) *Let W be an $\mathbb{F}$-structure on V and identify W^* with an $\mathbb{F}$-vector subspace of V^*. Assume that X is a finite set and let $p \in V^*$ such that $p(1_{\mathcal{Z}}) = 1$. If* $\operatorname{ev}_X V \subseteq W^*$ *and* $p \in (\operatorname{int} \operatorname{cone} \operatorname{ev}_X V) \cap W^*$, *then X is the support of a V-nondegenerate $\mathbb{F}$-weighted levelling space $\mathcal{X}$ such that* $\operatorname{centroid}_{\mathcal{X}} V = p$.
- (vi) *Assume that X is a finite set. If* $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$, *then X is the support of a finite V-nondegenerate weighted V-design $\mathcal{X}$ on $\mathcal{Z}$.*

Proof. (i) For every $x \in X$, we have $(\operatorname{ev}_x V)(1_{\mathcal{Z}}) = 1_{\mathcal{Z}}(x) = 1$. Therefore, $\sum_{i=1}^n c_i \operatorname{ev}_x V \in A$ if and only if $\sum_{i=1}^n c_i = 1$, hence the result.
(ii) Since V is finite dimensional, $\dim \operatorname{cone} \operatorname{ev}_X V = \dim V|_X$, hence the result.
(iii) We have

$$\begin{aligned}
\operatorname{centroid}_{\mathcal{Z}} V &= \operatorname{centroid}_{\mathcal{X}} V && \mathcal{X} \text{ is a } V\text{-design on } \mathcal{Z} \\
&\in \operatorname{reint} \operatorname{conv} \operatorname{ev}_X V && \text{[SZ84, Lemma 3.1] applied to } \mathcal{X} \\
&\subseteq \operatorname{reint} \operatorname{cone} \operatorname{ev}_X V && \text{(i)} \\
&= \operatorname{int} \operatorname{cone} \operatorname{ev}_X V. && \text{(ii)}
\end{aligned}$$

(iv) Let $p' := p/p(1_{\mathcal{Z}}) \in A$. By (i), p' is in the interior of $\operatorname{conv} \operatorname{ev}_X V$ in A. Applying Steinitz's theorem [GW93, Part 2.1, Theorem 10.3], we find a

finite set $Y \subseteq X$ such that p' is in the interior of $\operatorname{conv} \operatorname{ev}_Y V$ in A. Therefore, by (i) again, $p' \in \operatorname{int} \operatorname{cone} \operatorname{ev}_Y V$, hence $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_Y V$.
(v) Since $p \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$, by (ii), X is V-nondegenerate. By (i) and (ii), $p \in (\operatorname{int} \operatorname{cone} \operatorname{ev}_X V) \cap A = \operatorname{reint} \operatorname{conv} \operatorname{ev}_X V$, hence p can be written as a positive convex linear combination of points in $\operatorname{ev}_X V$, namely $p = \sum_{x \in X} c_x \operatorname{ev}_x V$ for some $c_x \in (0,1)$ such that $\sum_{x \in X} c_x = 1$. Since p and $\operatorname{ev}_X V$ are both in the $\mathbb{F}$-structure W^*, we may, in addition, assume that $c_x \in (0,1)_{\mathbb{F}}$ for every $x \in X$. Then the levelling space $\mathcal{X} := (X, \mu_X)$ with $\mu_X(x) := c_x$ satisfies the requirement $\operatorname{centroid}_{\mathcal{X}} V = p$.
(vi) It is clear that $\operatorname{centroid}_{\mathcal{Z}} 1_{\mathcal{Z}} = 1$. The result follows from (v) by taking $\mathbb{F} = \mathbb{R}$, $W = V$ and $p = \operatorname{centroid}_{\mathcal{Z}} V$. □

Proposition 2.15. *Assume $1_{\mathcal{Z}} \in V$. Then, there exists a finite V-nondegenerate weighted V-design on $\mathcal{Z}$.*

Proof. Applying Proposition 2.14(iii) to the V-nondegenerate weighted V-design $\mathcal{Z}$ itself on $\mathcal{Z}$, we get $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_Z V$. Then by Proposition 2.14(iv), there exists a finite subset $X \subseteq Z$ such that $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$. Therefore, by Proposition 2.14(vi), X is the support of a finite V-nondegenerate weighted V-design. □

Lemma 2.16. *Assume $1_{\mathcal{Z}} \in V$. Let $X \subseteq \mathcal{Z}$ be the support of a finite V-nondegenerate weighted V-design on $\mathcal{Z}$, and $S \subseteq \mathcal{Z}$ a finite subset. Then, $X \cup S$ is the support of a finite V-nondegenerate weighted V-design on $\mathcal{Z}$.*

Proof. By Proposition 2.14(iii),

$$\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V \subseteq \operatorname{int} \operatorname{cone} \operatorname{ev}_{X \cup S} V.$$

The result follows from Proposition 2.14(vi). □

Lemma 2.17. *Assume $1_{\mathcal{Z}} \in V$. Let $\mathcal{X} \subseteq \mathcal{Z}$ be a finite V-nondegenerate levelling space. For each point $x \in \mathcal{X}$, let $(x^{(i)} \in \mathcal{Z})_{i \in \mathbb{N}}$ be a sequence of points such that $\lim_{i \to \infty} \operatorname{ev}_{x^{(i)}} V = \operatorname{ev}_x V$ in V^*. Then, for all sufficiently large i, $X^{(i)} := \{x^{(i)} : x \in \mathcal{X}\}$ is the support of a finite V-nondegenerate levelling space $\mathcal{X}^{(i)}$ that is V-equivalent to $\mathcal{X}$.*

Proof. Since V^* is a finite dimensional real topological vector space, it is metrizable. It is then easy to see that $\operatorname{ev}_{X^{(i)}} V$ converges to $\operatorname{ev}_X V$ with respect to Hausdorff distance in V^*, hence $\operatorname{cone} \operatorname{ev}_{X^{(i)}} V$ converges to $\operatorname{cone} \operatorname{ev}_X V$ with respect to Hausdorff distance on compact supports in V^*. Since $\mathcal{X}$ is V-nondegenerate, for all sufficiently large i, by Proposition 2.14(ii), $\dim \operatorname{cone} \operatorname{ev}_{X^{(i)}} V = \dim \operatorname{cone} \operatorname{ev}_X V = \dim V$, which shows that $X^{(i)}$ is V-nondegenerate. Thus, $\operatorname{int} \operatorname{cone} \operatorname{ev}_{X^{(i)}} V$ converges to $\operatorname{int} \operatorname{cone} \operatorname{ev}_X V$ with respect to Hausdorff distance on compact supports in V^*. Applying Proposition 2.14(iii) to the V-nondegenerate weighted V-design $\mathcal{X}$ on $\mathcal{Z}$, we get $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_X V$. Therefore, for all sufficiently large i, $\operatorname{centroid}_{\mathcal{Z}} V \in \operatorname{int} \operatorname{cone} \operatorname{ev}_{X^{(i)}} V$, and the result follows from Proposition 2.14(vi). □

3. Weighted rational designs

In this section, we first define in § 3.1 rational designs and rational weighted designs in Euclidean spaces. We adopt Definition 3.1 for designs in the remainder of the paper unless explicitly stated otherwise. Then, we introduce two conditions on the spaces, Condition 3.2(i) and (ii), and prove in § 3.2 and 3.3 that they are necessary and sufficient for the existence of weighted rational designs.

3.1. **Designs in Euclidean spaces.** Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space. The algebra of all *polynomials on* $\mathcal{Z}$ is defined to be

$$\mathcal{P}[\mathcal{Z}] := \mathbb{R}[x_1, \dots, x_d]/I(\mathcal{Z}),$$

where $I(\mathcal{Z})$ is the ideal of polynomials in $\mathbb{R}[x_1, \dots, x_d]$ vanishing on $\mathcal{Z}$. For an ideal I in $\mathbb{R}[x_1, \dots, x_d]$, let $I_{\leq t}$ be the vector subspace of I consisting of polynomials of degree bounded above by t. The algebra $\mathcal{P}[\mathcal{Z}]$ admits a filtration of vector spaces:

$$0 \subseteq \mathcal{P}^0[\mathcal{Z}] \subseteq \mathcal{P}^1[\mathcal{Z}] \subseteq \cdots,$$

where

$$\mathcal{P}^t[\mathcal{Z}] := \mathbb{R}[x_1, \dots, x_d]_{\leq t}/I(\mathcal{Z})_{\leq t}$$

is a vector subspace of $\mathcal{P}[\mathcal{Z}]$. In other words, $\mathcal{P}^t[\mathcal{Z}]$ is the vector subspace of $\mathcal{P}[\mathcal{Z}]$ generated by polynomials of degree at most t.

In particular, on the real unit d-sphere $\mathcal{S}^d$,

$$\mathcal{P}^t[\mathcal{S}^d] = \mathbb{R}[x_0, \dots, x_d]_{\leq t}/(x_0^2 + \cdots + x_d^2 - 1)_{\leq t},$$

and on the open unit interval $\mathcal{I}$,

$$\mathcal{P}^t[\mathcal{I}] = \mathbb{R}[x]_{\leq t}.$$

Definition 3.1. Suppose that all polynomials on $\mathcal{Z}$ are integrable on $\mathcal{Z}$. A *t-design* (resp. *weighted t-design*) $\mathcal{X}$ on $\mathcal{Z}$ is a $\mathcal{P}^t[\mathcal{Z}]$-design (resp. weighted $\mathcal{P}^t[\mathcal{Z}]$-design) on $\mathcal{Z}$ (as introduced in Definition 2.11). A (weighted) t-design $\mathcal{X}$ on $\mathcal{Z}$ is called *rational* if it consists of rational points, namely $\mathcal{X} \subseteq \mathbb{Q}^d$.

We say that there are *enough designs on* $\mathcal{Z}$ satisfying some given additional properties provided that there are t-designs on $\mathcal{Z}$ satisfying the properties for each natural number t. For example, by saying that there are enough rational designs on $\mathcal{Z}$, we mean that there are rational t-designs on $\mathcal{Z}$ for each natural number t.

Condition 3.2 below is a necessary and sufficient condition for the existence of enough rational designs on $\mathcal{Z}$, which we prove in § 3.2 and 3.3.

Condition 3.2. (i) The rational points $\mathcal{Z} \cap \mathbb{Q}^d$ in $\mathcal{Z}$ are dense in $\mathcal{Z}$.
(ii) For every polynomial $f \in \mathcal{P}[\mathcal{Z}]$ with rational coefficients, $\frac{1}{|\mathcal{Z}|} \int_Z f \,\mathrm{d}\,\mu_Z$ is a rational number.

3.2. **Necessity and sufficiency of Condition 3.2(i).** Recall that in Definition 2.7 we define the *centroid of a function f on $\mathcal{X}$* to be $\operatorname{centroid}_{\mathcal{X}} f := \frac{1}{|\mathcal{X}|}\int_X f \,\mathrm{d}\,\mu_X$.

Proposition 3.3. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a bounded levelling space, and $Y \subseteq \mathcal{Z}$ a subset. If there are enough weighted designs on $\mathcal{Z}$ consisting of points in Y, then Y is dense in $\mathcal{Z}$. In particular, Condition 3.2(i) is a necessary condition for $\mathcal{Z}$ to have enough rational designs.*

Proof. Let p be an arbitrary point in $\mathcal{Z}$. For each $e \in \mathbb{N}$, let $\mathcal{X}_{2e}$ be a weighted $2e$-design on $\mathcal{Z}$ consisting of points in Y. Since $\mathcal{Z}$ is bounded, there exists a real number d that is larger than the diameter of $\mathcal{Z}$. Then, the quadratic polynomial f_p on $\mathcal{Z}$ given by

$$f_p(x) := 1 - \frac{1}{d^2}\|x-p\|_2^2$$

has the property that $f_p(\mathcal{Z}) \subseteq (0,1]$ and f_p achieves the maximal value 1 at the point p. For $\epsilon \in (0,1]$, let p_ϵ denote the preimage $f_p^{-1}((1-\epsilon,1])$, which is an open set in $\mathcal{Z}$ and has positive measure.

Suppose that Y is not dense around p. Let $\delta := 1 - \sup f_p(Y) > 0$, and let $\epsilon \in (0,\delta)$ be a fixed number. Since f_p^e is of degree $2e$ and $\mathcal{X}_{2e}$ is a weighted $2e$-design,

$$(1-\delta)^e \geq \operatorname{centroid}_{\mathcal{X}_{2e}} f_p^e = \operatorname{centroid}_{\mathcal{Z}} f_p^e \geq \frac{\mu_Z(p_\epsilon)}{|\mathcal{Z}|}(1-\epsilon)^e,$$

where $\mu_Z(p_\epsilon) > 0$. Taking $e \to \infty$, we get a contradiction. Therefore, Y is dense around the arbitrarily chosen point $p \in \mathcal{Z}$. □

Lemma 3.4. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space where Condition 3.2(i) holds and polynomials are integrable on $\mathcal{Z}$. Then, for every $t \in \mathbb{N}$, there exists a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design on $\mathcal{Z}$.*

Proof. Applying Proposition 2.15 with $V = \mathcal{P}^t[\mathcal{Z}]$, we get a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted t-design $\mathcal{X}$ on $\mathcal{Z}$. For each point $x \in \mathcal{X}$, we pick a sequence of rational points $(x^{(i)} \in \mathcal{Z} \cap \mathbb{Q}^d)_{i\in\mathbb{N}}$ whose limit is x. By Lemma 2.17, we know that for some sufficiently large i, $X^{(i)} := \{x^{(i)} : x \in \mathcal{X}\}$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate levelling space $\mathcal{X}^{(i)}$ that is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\mathcal{X}$. Then, $\mathcal{X}^{(i)}$ is a desired design. □

3.3. **Necessity and sufficiency of Condition 3.2(ii).** Let $k \subseteq \mathbb{R}$ be a subset. Recall that in Definition 2.2, we say that a levelling space $\mathcal{X}$ is *k-weighted* if μ_X takes values in k.

Proposition 3.5. *Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield, and let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space. If there are enough finite $\mathbb{F}$-weighted rational designs on $\mathcal{Z}$, then* $\operatorname{centroid}_{\mathcal{Z}} f \in \mathbb{F}$ *for every monic monomial f in $\mathcal{P}[\mathcal{Z}]$. In particular, Condition 3.2(ii) is a necessary condition for having enough $\mathbb{Q}$-weighted rational designs on $\mathcal{Z}$.*

Proof. Let t be a natural number and $\mathcal{X}$ a finite $\mathbb{F}$-weighted rational t-design on $\mathcal{Z}$. By the definition of weighted designs, for every monic monomial f in $\mathcal{P}^t[\mathcal{Z}]$,

$$\operatorname{centroid}_{\mathcal{Z}} f = \operatorname{centroid}_{\mathcal{X}} f = \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \mu_X(x) f(x),$$

which is an $\mathbb{F}$-linear combination of rational numbers $\{f(x) : x \in \mathcal{X}\}$. Therefore, $\operatorname{centroid}_{\mathcal{Z}} f \in \mathbb{F}$. Since the choice of t is arbitrary, the result holds for all monic monomials $f \in \bigcup_{t\in\mathbb{N}} \mathcal{P}^t[\mathcal{Z}] = \mathcal{P}[\mathcal{Z}]$. $\square$

Lemma 3.6. *Let $\mathbb{F} \subseteq \mathbb{R}$ be a subfield and $\mathcal{Z} \subseteq \mathbb{R}^d$ a levelling space. Suppose that $X \subseteq \mathcal{Z} \cap \mathbb{Q}^d$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted levelling space $\mathcal{X}$ such that* $\operatorname{centroid}_{\mathcal{X}} f \in \mathbb{F}$ *for all monic monomials $f \in \mathcal{P}^t[\mathcal{Z}]$. Then, X is also the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate $\mathbb{F}$-weighted levelling space $\widetilde{\mathcal{X}}$ that is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\mathcal{X}$. Moreover, when $\mathbb{F} = \mathbb{Q}$, $\widetilde{\mathcal{X}}$ can be taken to be integer-weighted.*

Proof. Choose a basis of $\mathcal{P}^t[\mathcal{Z}]$ consisting of monic monomials, and let $\mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]$ be the $\mathbb{F}$-vector space generated by the basis. Let $p := \operatorname{centroid}_{\mathcal{X}} \mathcal{P}^t[\mathcal{Z}] \in \mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]^*$. By Proposition 2.14(iii), $p \in \operatorname{int}\operatorname{cone}\operatorname{ev}_X \mathcal{P}^t[\mathcal{Z}]$. Since X consists of rational points, $\operatorname{ev}_X \mathcal{P}^t[\mathcal{Z}] \subseteq \mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]^*$. Applying Proposition 2.14(v) to the $\mathbb{F}$-structure $\mathcal{P}^t_{\mathbb{F}}[\mathcal{Z}]$ of $\mathcal{P}^t[\mathcal{Z}]$, point p and finite set X, we get a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate $\mathbb{F}$-weighted levelling space $\widetilde{\mathcal{X}}$ with support X that is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\mathcal{X}$. When $\mathbb{F} = \mathbb{Q}$, a suitable integer scalar $n\widetilde{\mathcal{X}}$ is an integer-weighted levelling space, which is $\mathcal{P}^t[\mathcal{Z}]$-equivalent to $\widetilde{\mathcal{X}}$ by Lemma 2.9. $\square$

Lemma 3.7. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space satisfying Condition 3.2(ii). Suppose that $\mathcal{X} \subseteq \mathcal{Z}$ is a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design on $\mathcal{Z}$ with support X. Then, X is the support of a $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate integer-weighted rational t-design $\widetilde{\mathcal{X}}$ on $\mathcal{Z}$.*

Proof. Since $\operatorname{centroid}_{\mathcal{X}} \mathcal{P}^t[\mathcal{Z}] = \operatorname{centroid}_{\mathcal{Z}} \mathcal{P}^t[\mathcal{Z}]$, applying Lemma 3.6 to $\mathbb{F} = \mathbb{Q}$ and $\mathcal{X}$, we get a desired t-design $\widetilde{\mathcal{X}}$. $\square$

The following proposition is stronger than Lemma 3.4. It will not be used in the proof of Theorem 7.1 as Lemma 3.4 suffices for our purpose.

Proposition 3.8. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space satisfying Condition 3.2. Assume that $\mathcal{Z}$ is not finite. Then, for every $t \in \mathbb{N}$ and every sufficiently large $n \in \mathbb{N}$, there exists a $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate integer-weighted rational t-design $\mathcal{X}$ on $\mathcal{Z}$ of size n.*

Proof. By Lemma 3.4, there exists a finite weighted $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate rational t-design $\mathcal{X}$ on $\mathcal{Z}$. Let $S \subseteq \mathcal{Z} \cap \mathbb{Q}^d$ be a finite subset such that $\#(X \cup S) = n$. By Lemma 2.16, $X \cup S$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design of size n. Then, according to Lemma 3.7, $X \cup S$ is the support of a desired design. $\square$

4. Total measures of integer-weighted designs

The main result in this section is Theorem 4.5, which shows that for every sufficiently large total measure, we can find an integer-weighted levelling space with the given total measure such that it is equivalent to a fixed levelling space. Theorem 4.5 is used in the proof of Theorem 7.1.

4.1. **Vandermonde matrix and its p-adic valuation.** For a prime number p, let ν_p be the p-adic valuation. In other words, for $n \in \mathbb{Z}$, $\nu_p(n) := \sup\{v \in \mathbb{N} : \ p^v | n\}$, and for $a/b \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$, $\nu_p(a/b) := \nu_p(a) - \nu_p(b)$. For a matrix $\mathbf{A}$ over $\mathbb{Q}$, let $\nu_p(\mathbf{A}) := \min\{\nu_p(a) : \ \text{entry } a \text{ of } \mathbf{A}\}$. For two matrices $\mathbf{A}$ and $\mathbf{A}'$ over $\mathbb{Q}$, we have $\nu_p(\mathbf{A}\mathbf{A}') \geq \nu_p(\mathbf{A}) + \nu_p(\mathbf{A}')$ whenever the sizes of $\mathbf{A}$ and $\mathbf{A}'$ are compatible for matrix multiplication.

Definition 4.1. Let t and n be two natural numbers, and let $\mathbf{a} := (a_1, \dots, a_n)$ be a tuple of rational numbers. The *t-th Vandermonde matrix of* $\mathbf{a}$ is the matrix $\mathbf{A}$ whose rows are indexed by $[0, t]_{\mathbb{Z}}$, columns indexed by $[1, n]_{\mathbb{Z}}$ and (i, j)-th entry defined to be $\mathbf{A}_{i,j} := a_j^i$, where $i \in [0, t]_{\mathbb{Z}}$ and $j \in [1, n]_{\mathbb{Z}}$.

Lemma 4.2. *Let n be a natural number, $\mathbf{a} := (a_1, \dots, a_n)$ a tuple of rational numbers and $\mathbf{A}$ the $(n-1)$-th Vandermonde matrix of $\mathbf{a}$, which is a square matrix. Suppose that p is a prime number such that $\nu_p(a_i)$'s are distinct negative integers where i runs over $[1, n]_{\mathbb{Z}}$. Then, for $i \in [1, n]_{\mathbb{Z}}$ and $j \in [0, n-1]_{\mathbb{Z}}$, we have $\nu_p((\mathbf{A}^{-1})_{i,j}) \geq -j\nu_p(a_i)$. In particular, $\nu_p((\mathbf{A}^{-1})_{i,j}) \geq j\nu_p(1/\mathbf{a})$, where $1/\mathbf{a} := (1/a_1, \dots, 1/a_n)$.*

Proof. It is well-known that the (i, j)-th entry of $\mathbf{A}^{-1}$ is

$$(\mathbf{A}^{-1})_{i,j} = (-1)^j \frac{e_{n-j-1}(a_1, \dots, \hat{a}_i, \dots, a_n)}{\prod\limits_{\substack{k \in [1,n]_{\mathbb{Z}} \\ k \neq i}} (a_k - a_i)},$$

where $e_{n-j-1}(a_1, \dots, \hat{a}_i, \dots, a_n)$ is the $(n-j-1)$-th elementary symmetric polynomial. Since all $\nu_p(a_i)$'s are distinct, it is straightforward to calculate the p-adic valuation of the numerators and denominators of $(\mathbf{A}^{-1})_{i,j}$, and verify that $\nu_p((\mathbf{A}^{-1})_{i,j}) \geq -j\nu_p(a_i)$. $\square$

Lemma 4.3. *Let t and n be two natural numbers such that $n \geq t+1$, $\mathbf{a} := (a_1, \dots, a_n)$ a tuple of positive rational numbers, $\mathbf{A}$ the t-th Vandermonde matrix of $\mathbf{a}$, and $\mathbf{b} := (b_0, \dots, b_t) \in \mathbb{Q}^{t+1}$ a column vector. Suppose that p is a prime number such that $\nu_p(a_i)$'s are distinct negative integers where i runs over $[1, n]_{\mathbb{Z}}$. Assume that the linear system*

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{4.1}$$

has a positive real solution $\mathbf{x} \in \mathbb{R}^n_{>0}$. Then, Eq. (4.1) has a positive rational solution $\mathbf{y} \in \mathbb{Q}^n_{>0}$ such that $\nu_p(\mathbf{y}) \geq \min\{j\nu_p(1/\mathbf{a}) + \nu_p(b_j) : \ j \in [0, t]_{\mathbb{Z}}\}$. In particular, if $\nu_p(1/\mathbf{a})$ is sufficiently large, then $\nu_p(\mathbf{y}) \geq \nu_p(b_0)$.

Proof. For a subset $S \subseteq [1,n]_{\mathbb{Z}}$, denote its complement by $\overline{S} := [1,n]_{\mathbb{Z}} \setminus S$, and let $\cdot|_S$ and $\cdot|_{\overline{S}}$ be the restriction maps to S and $\overline{S}$, respectively.

Since Eq. (4.1) has a positive real solution, there exists a positive rational number c such that $\nu_p(c)$ is sufficiently large and that the system

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq c\mathbf{1}, \end{cases} \tag{4.2}$$

has a real solution with $\mathbf{x} > c\mathbf{1}$. Clearly, the dimension of the solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $n-t-1$. Moreover, due to the existence of a solution $\mathbf{x} > c\mathbf{1}$, the convex polytope defined by Eq. (4.2) also has dimension $n-t-1$. Let $\mathbf{y}$ be an extreme point of this convex polytope. We then know that $\mathbf{y}$ can be uniquely determined by the intersection of n hyperplanes, $t+1$ of which are given by $\mathbf{A}\mathbf{x} = \mathbf{b}$, and the other $n-t-1$ are of the form $x_i = c$ where $i \in \overline{S}$ for some subset $S \subseteq [1,n]_{\mathbb{Z}}$ of size $t+1$. It is then easy to see from Eq. (4.2) that $\mathbf{A}|_S \cdot \mathbf{y}|_S + \mathbf{A}|_{\overline{S}} \cdot \mathbf{y}|_{\overline{S}} = \mathbf{b}$. Since $\mathbf{y}$ is an extreme point, $\mathbf{A}|_S$ is an invertible Vandermonde matrix and $\mathbf{y}$ is a rational point whose coordinates are given by

$$\begin{cases} \mathbf{y}|_S = \mathbf{A}|_S^{-1} \cdot (\mathbf{b} - \mathbf{A}|_{\overline{S}} \cdot \mathbf{y}|_{\overline{S}}), \\ \mathbf{y}|_{\overline{S}} = c\mathbf{1}. \end{cases} \tag{4.3}$$

Thus,

$$\begin{aligned} \nu_p(\mathbf{y}) &= \min\{\nu_p(\mathbf{y}|_S), \nu_p(\mathbf{y}|_{\overline{S}})\} \\ &\geq \min\big\{\nu_p(\mathbf{A}|_S^{-1} \cdot \mathbf{b}), \\ &\quad \nu_p(\mathbf{A}|_S^{-1}) + \nu_p(\mathbf{A}|_{\overline{S}}) + \nu_p(c), \nu_p(c)\big\} && \text{Eq. (4.3)} \\ &= \nu_p(\mathbf{A}|_S^{-1} \cdot \mathbf{b}) && \nu_p(c) \text{ is sufficiently large} \\ &\geq \min\{j\nu_p(1/\mathbf{a}|_S) + \nu_p(b_j) : \ j \in [0,t]_{\mathbb{Z}}\}, && \text{Lemma 4.2 applied to } \mathbf{A}|_S \\ &\geq \min\{j\nu_p(1/\mathbf{a}) + \nu_p(b_j) : \ j \in [0,t]_{\mathbb{Z}}\}, && \nu_p(1/\mathbf{a}|_S) \geq \nu_p(1/\mathbf{a}) \end{aligned}$$

which completes the proof. □

4.2. **Realization of total measures.**

Proposition 4.4. *Let t be a natural number and let $\mathcal{X} \subseteq \mathcal{I} \cap \mathbb{Q}$ be a finite $\mathcal{P}^t[\mathcal{I}]$-nondegenerate levelling space. For every prime number p, there exists an integer-weighted levelling space $\mathcal{X}_p \subseteq \mathcal{I} \cap \mathbb{Q}$ such that $\mathcal{X}_p$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$ and the total measure of $\mathcal{X}_p$ is not divisible by p.*

Proof. Let $\mathbf{b} := (b_j : \ j \in [0,t]_{\mathbb{Z}})$, where $b_j := \operatorname{centroid}_{\mathcal{X}} f_j$ and $f_j(x) := x^j$. It is clear that $b_0 = 1$. For each $x \in \mathcal{X}$, we choose a rational point $a_x \in \mathcal{I} \cap \mathbb{Q}$ sufficiently close to x while $\nu_p(1/a_x)$'s are sufficiently large distinct natural numbers. Let $\overline{\mathbf{a}} := \{a_x : \ x \in \mathcal{X}\}$, $\mathbf{a} := (a_x : \ x \in \mathcal{X})$, and let $\mathbf{A}$ be the t-th Vandermonde matrix of $\mathbf{a}$.

According to Lemma 2.17, $\overline{\mathbf{a}}$ is the support of a finite levelling space $\mathcal{A}$ that is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$. So, Eq. (4.1) has a positive real solution

$\mathbf{u} := (\mu_A(a_x)/|\mathcal{A}| : \ x \in \mathcal{X})$. Then, by Lemma 4.3, Eq. (4.1) also has a positive rational solution $\mathbf{w} = (w_x : \ x \in \mathcal{X})$ such that $\nu_p(\mathbf{w}) \geq \nu_p(b_0) = 0$. The solution $\mathbf{w}$ gives us a rational-weighted levelling space $\mathcal{W}$ with support $\bar{\mathbf{a}}$, measure $\mu_W(a_x) = w_x$ for all $x \in \mathcal{X}$ and total measure $\sum_{x\in\mathcal{X}} w_x = b_0 = 1$. The levelling space $\mathcal{W}$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{A}$ since $\mathbf{Aw} = \mathbf{Au}$.

Let m be the least common multiple of the denominators of the rational coordinates of $\mathbf{w}$. Since $\nu_p(\mathbf{w}) \geq 0$, m is not divisible by p. Therefore, $m\,\mathcal{W}$ is an integer-weighted levelling space, which is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{W}$ by Lemma 2.9, whose total measure is m, which is not divisible by p. □

Theorem 4.5. *Let t be a natural number and let $\mathcal{X} \subseteq \mathcal{I} \cap \mathbb{Q}$ be a $\mathcal{P}^t[\mathcal{I}]$-nondegenerate integer-weighted levelling space. Then, there exists a natural number n_0 such that for every natural number $n \geq n_0$, there exists an integer-weighted levelling space $\mathcal{X}_n \subseteq \mathcal{I} \cap \mathbb{Q}$ with total measure n such that $\mathcal{X}_n$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$.*

Proof. Let P be the set of all prime factors of the total measure $|\,\mathcal{X}\,|$. For each $p \in P$, we apply Proposition 4.4 and get an integer-weighted levelling space $\mathcal{X}_p \subseteq \mathcal{I} \cap \mathbb{Q}$ with support X_p that is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$ such that $|\,\mathcal{X}_p\,|$ is not divisible by p. Therefore, the additive semigroup generated by $|\,\mathcal{X}\,|$ and $|\,\mathcal{X}_p\,|$ for all $p \in P$, which is a numerical semigroup, contains all sufficiently large natural numbers. Let n be an arbitrary sufficiently large natural number. Then, it can be written as a finite linear combination

$$n = c_0|\,\mathcal{X}\,| + \sum_{p\in P} c_p|\,\mathcal{X}_p\,|$$

for some natural numbers c_0 and c_p. By Lemma 2.9,

$$\mathcal{X}_n := c_0\,\mathcal{X} + \sum_{p\in P} c_p\,\mathcal{X}_p$$

is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$. Moreover, $\mathcal{X}_n$ is integer-weighted and has total measure n. □

5. Hilbert-Kamke problem

We present in this section some results on the Hilbert-Kamke problem. Our main result in this section is Theorem 5.3, which is used in the proof of Theorem 7.1.

Let t and n be fixed natural numbers and let $\mathbf{c} := (c_1, \ldots, c_t)$ be a fixed tuple of rational numbers. Consider the Hilbert-Kamke type system of t equations in n variables x_i's:

$$\sum_{i=1}^{n} x_i^k = c_k, \quad k \in [1,t]_{\mathbb{Z}}. \tag{5.1}$$

For a positive integer P, let $\mathrm{J}(t, n; \mathbf{c}; P)$ denote the set of all solutions of Eq. (5.1) such that $x_i \in (0,1)_{P^{-1}\mathbb{Z}}$ for all $i \in [1,n]_{\mathbb{Z}}$, where $P^{-1}\mathbb{Z} := \{P^{-1}q : \ q \in \mathbb{Z}\}$ is a fractional ideal over $\mathbb{Z}$.

Remark 5.1. Let $P_{\mathbf{c}}$ be the smallest positive integer P such that $c_k \in P^{-k}\mathbb{Z}$ for every integer $k \in [1,t]_{\mathbb{Z}}$. Then $\mathrm{J}(t,n;\mathbf{c};P) = \emptyset$ unless P is a positive integer multiple of $P_{\mathbf{c}}$.

The study of $\mathrm{J}(t,n;\mathbf{c};P)$ is a variation of the famous Hilbert-Kamke problem, which has been addressed over the last several decades [Mar53; Ark85; Woo12]. For our purpose, we need to show the existence of solutions in $\mathrm{J}(t,n;\mathbf{c};P)$ satisfying some additional properties, which is obtained through an asymptotic formula for the size of $\mathrm{J}(t,n;\mathbf{c};P)$ when P goes to infinity in $P_{\mathbf{c}}\mathbb{Z}_{>0}$. We reformulate below in Theorem 5.2 the asymptotic results in [Ark85] to suit our purpose.

We use Vinogradov's notation for asymptotic formulas in this paper. Let f and g be two functions on a domain $D \subseteq \mathbb{R}$, and let $y_1,\dots,y_n$ be some objects. Denote by $f \ll_{y_1,\dots,y_n} g$ the fact that there exists a positive constant $c_{y_1,\dots,y_n}$ that only depends on $y_1,\dots,y_n$ such that $|f(x)| \le c_{y_1,\dots,y_n}|g(x)|$ for all sufficiently large $x \in D$. We write $f \gg_{y_1,\dots,y_n} g$ if $g \ll_{y_1,\dots,y_n} f$, and write $f \asymp_{y_1,\dots,y_n} g$ if $f \ll_{y_1,\dots,y_n} g \ll_{y_1,\dots,y_n} f$.

Theorem 5.2. *Let t, n, $\mathbf{c}$ and $P_{\mathbf{c}}$ be fixed as above. Suppose that $n \ge 3t^2 2^t - t$ and consider the domain $D := \{P \in P_{\mathbf{c}}\mathbb{Z}_{>0} : P \ge n^{10}\}$. Then there exist real-valued functions σ, γ and θ of P such that the following statements hold.*

(i) *On the domain D,*

$$\#\mathrm{J}(t,n;\mathbf{c};P) = \sigma\cdot\gamma\cdot(P-1)^{n-t(t+1)/2} + \theta\cdot n^{30n^3}(P-1)^{n-t(t+1)/2-1/30(2+\log t)}.$$

(ii) *For all $P \ge 1$,*

$$\#\mathrm{J}(t,n;\mathbf{c};P) \le n^{30n^3}(P-1)^{n-t(t+1)/2}.$$

(iii) *On the domain D,*

$$|\theta| < 1.$$

(iv) *If there exists a p-adic solution $\mathbf{y} \in (P^{-1}\mathbb{Z}_p)^n \subseteq \mathbb{Q}_p^n$ to Eq. (5.1) for each prime number p, then on the domain D,*

$$\sigma \ge n^{-20n^4 2^n}.$$

(v) *If there exists a real solution $\mathbf{y} = (y_1,\dots,y_n) \in [P^{-1}, 1-P^{-1}]^n$ to Eq. (5.1), then on the domain D,*

$$2^{2t(t-n)}n^{(t-n)}t^{-n-t}(\Delta\mathbf{y})^{t(n-t)} \le \gamma \le 2^{2t^2}n^{2t}t^{n-2t}(t+1)^{3t-n},$$

where

$$\Delta\mathbf{y} := \max_{z_0,\dots,z_{t+1}\in\{0,y_1,\dots,y_n,1\}}\left(\min_{0\le i<j\le t+1}|z_i - z_j|\right).$$

(vi) *If there exists a rational solution $\mathbf{y} = (y_1,\dots,y_n) \in (0,1)_{\mathbb{Q}}^n$ to Eq. (5.1) such that the number of distinct elements of $y_1,\dots,y_n$ is at least t, then*

$$\#\mathrm{J}(t,n;\mathbf{c};P) \asymp_{t,n,\mathbf{y}} P^{n-t(t+1)/2}$$

as $P \to \infty$ in the domain D.

Proof. Although [Ark85] assumes that $t \geq 3$, the proof there also works for $t \in \{0, 1, 2\}$. So, when we refer to theorems in [Ark85], such restriction on t will be omitted.

Let $N_k := c_k P^k$ for all $k \in [1, t]_{\mathbb{Z}}$. It is clear that there exists a bijection sending a solution $(y_1, \dots, y_n) \in (0, 1)^n_{P^{-1}\mathbb{Z}}$ of Eq. (5.1) to a solution $(Py_1, \dots, Py_n) \in [1, P-1]^n_{\mathbb{Z}}$ of

$$\sum_{i=1}^{n} x_i^k = N_k, \quad k \in [1, t]_{\mathbb{Z}}.$$

Now we consider the counting function $J(t, n; N_1, \dots, N_t; P-1)$ appearing in [Ark85, Theorem 1]. Then, by definition,

$$\#\mathrm{J}(t, n; \mathbf{c}; P) = J(t, n; N_1, \dots, N_t; P-1).$$

Let

$$\sigma := \sigma(t, n; P-1)$$

be the sum of the singular series defined in [Ark85, Theorem 1] and

$$\gamma := \gamma(t, n; N_1, \dots N_t; P-1)$$

be the value of the singular integral defined in [Ark85, Theorem 1].
(i), (ii), (iii): The result follows from [Ark85, Theorem 1].
(iv): The result follows from [Ark85, Theorem 3, 4].
(v): The result follows from [Ark85, Theorem 5].
(vi): As discussed in Remark 5.1, the rational solution $\mathbf{y}$ must be in $(0, 1)^n_{P^{-1}\mathbb{Z}}$ for some $P \in D$. Then, the solution $\mathbf{y}$ is automatically a p-adic solution in $(P^{-1}\mathbb{Z}_p)^n$ for each prime p and a real solution in $[P^{-1}, 1-P^{-1}]^n$. Since there exist at least t distinct elements in $y_1, \dots, y_n$, we know that $\Delta\mathbf{y} > 0$. Therefore by (iv) and (v), $\sigma, \gamma \gg_{t,n,\mathbf{y}} 1$. Combining the statements (i), (ii) and (iii) yields (vi). $\square$

Using the asymptotic formula from Theorem 5.2, we count in Theorem 5.3 the number of solutions such that x_i's are not the same in a certain sense, and formulate the result in the language of levelling spaces. Recall that $\mathcal{I} = (0, 1)$ is the open interval equipped with the Lebesgue measure.

Theorem 5.3. *Let t be a natural number, and let $\mathcal{X} \subseteq \mathcal{I} \cap \mathbb{Q}$ be an integer-weighted levelling space. Suppose that $\# \mathcal{X} \geq t$ and $|\,\mathcal{X}\,| \geq 3t^2 2^t - t$. Let $p \in \mathbb{Q}[x]^d$ be a nonconstant polynomial map $\mathbb{Q} \to \mathbb{Q}^d$ for some positive integer d. For a positive integer P, let $\mathrm{J}_p(t; \mathcal{X}; P)$ (resp. $\widetilde{\mathrm{J}}_p(t; \mathcal{X}; P)$) denote the set of all integer-weighted levelling spaces $\mathcal{Y} = (Y, \mu_Y)$, such that μ_Y is the counting measure (resp. μ_Y is not the counting measure, i.e. $\mu_Y(y) \geq 2$ for some $y \in Y$) and that:*

(i) *$\mathcal{Y} \subseteq \mathcal{I} \cap P^{-1}\mathbb{Z}$;*
(ii) *$|\,\mathcal{Y}\,| = |\,\mathcal{X}\,|$;*
(iii) *$\mathcal{Y}$ is $\mathcal{P}^t[\mathcal{I}]$-equivalent to $\mathcal{X}$;*

(iv) *p is injective on Y.*

Then,

$$\#\mathrm{J}_p(t;\mathcal{X};P) \asymp_{t,\mathcal{X},p} P^{|\,\mathcal{X}\,|-t(t+1)/2} \tag{5.2}$$

and

$$\#\widetilde{\mathrm{J}}_p(t;\mathcal{X};P) \ll_{t,\mathcal{X},p} P^{|\,\mathcal{X}\,|-1-t(t+1)/2} \tag{5.3}$$

as $P \to \infty$ in the domain $P_0\,\mathbb{Z}_{>0}$ for some positive integer P_0 which depends only on t and $\mathcal{X}$. For $P \notin P_0\,\mathbb{Z}_{>0}$, both $\mathrm{J}_p(t;\mathcal{X};P)$ and $\widetilde{\mathrm{J}}_p(t;\mathcal{X};P)$ are empty.

Proof. The set of monic monomials $\{f_k : \ k \in [0,t]_{\mathbb{Z}}\}$, where $f_k(x) := x^k$, forms a basis of $\mathcal{P}^t[\mathcal{I}]$, hence condition (iii) means that

$$\operatorname{centroid}_{\mathcal{Y}} f_k = \operatorname{centroid}_{\mathcal{X}} f_k, \tag{5.4}$$

for all $k \in [0,t]_{\mathbb{Z}}$.

Let $\mathbf{c} := (c_1,\dots,c_t)$ where $c_k := |\,\mathcal{X}\,|\operatorname{centroid}_{\mathcal{X}} f_k$ for $k \in [1,t]_{\mathbb{Z}}$. Let $P_0 := P_{\mathbf{c}}$ be the positive integer defined in Remark 5.1. Then, both $\mathrm{J}_p(t;\mathcal{X};P)$ and $\widetilde{\mathrm{J}}_p(t;\mathcal{X};P)$ are empty when $P \notin P_0\,\mathbb{Z}_{>0}$. From now on, P is assumed to be in $P_0\,\mathbb{Z}_{>0}$

Let $n := |\,\mathcal{X}\,|$ be the total measure of $\mathcal{X}$. In this proof, we identify an integer-weighted levelling space $\mathcal{Y}$ with a tuple $\mathbf{y} = (y_1,\dots,y_n)$ where $y_1 \le \cdots \le y_n$, such that y_i's are elements in $\mathcal{Y}$ and each $y \in \mathcal{Y}$ appears $\mu_Y(y)$ times in $\mathbf{y}$. Clearly, Eq. (5.4) holds for $k=0$. For each $k \in [1,t]_{\mathbb{Z}}$, Eq. (5.4) holds for a levelling space $\mathcal{Y}$ satisfying conditions (i) and (ii), if and only if

$$\sum_{i=1}^{n} y_i^k = c_k,$$

since $\operatorname{centroid}_{\mathcal{Y}} f_k = \sum_{i=1}^n y_i^k/n$ and $\operatorname{centroid}_{\mathcal{X}} f_k = c_k/n$. Let $\mathrm{J}(t;\mathcal{X};P)$ be the set of levelling spaces $\mathcal{Y}$ satisfying conditions (i), (ii) and (iii). Then, there exists a bijection between $\mathrm{J}(t;\mathcal{X};P)$ and $\mathrm{J}(t,n;\mathbf{c};P)$ up to permutations of elements of a solution $(y_1,\dots,y_n)$.

Denote by $\mathrm{J}_p(t,n;\mathbf{c};P)$ the set of all solutions $(y_1,\dots,y_n)$ in $\mathrm{J}(t,n;\mathbf{c};P)$ such that all $p(y_i)$'s are distinct for $i \in [1,n]_{\mathbb{Z}}$, and let $\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)$ be the complement $\mathrm{J}(t,n;\mathbf{c};P) \setminus \mathrm{J}_p(t,n;\mathbf{c};P)$.

A levelling space $\mathcal{Y} \in \mathrm{J}(t;\mathcal{X};P)$ is in $\mathrm{J}_p(t;\mathcal{X};P)$ if and only if $p(y_i)$'s are distinct for $i \in [1,n]_{\mathbb{Z}}$, hence there exists a bijection between $\mathrm{J}_p(t;\mathcal{X};P)$ and $\mathrm{J}_p(t,n;\mathbf{c};P)$ up to permutations of elements of a solution. Therefore,

$$\#\mathrm{J}_p(t;\mathcal{X};P) \asymp_n \#\mathrm{J}_p(t,n;\mathbf{c};P). \tag{5.5}$$

For each levelling space $\mathcal{Y} \in \widetilde{\mathrm{J}}_p(t;\mathcal{X};P)$, we have $y_i = y_j$ for some indexes i,j because μ_Y is not the counting measure, hence $\mathbf{y} \in \widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P)$. Therefore,

$$\#\widetilde{\mathrm{J}}_p(t;\mathcal{X};P) \le \#\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P). \tag{5.6}$$

By Eq. (5.5) and Theorem 5.2(ii),

$$\#\mathrm{J}_p(t;\mathcal{X};P) \asymp_n \#\mathrm{J}_p(t,n;\mathbf{c};P) \le \#\mathrm{J}(t,n;\mathbf{c};P) \ll_n P^{n-t(t+1)/2},$$

which proves one direction of Eq. (5.2). On the other hand, for every solution $(y_1,\dots,y_n)$ of Eq. (5.1) with $p(y_u)=p(y_v)$ for some distinct $u,v\in[1,n]_{\mathbb{Z}}$, $(y_i:\ i\neq u,v)$ is a solution of the system of equations

$$\sum_{i=1}^{n-2} x_i^k = (\mathbf{c}_{y_u,y_v})_k, \quad k\in[1,t]_{\mathbb{Z}},$$

where $\mathbf{c}_{y_u,y_v} := (c_k - y_u^k - y_v^k:\ k\in[1,t]_{\mathbb{Z}})$. Let

$$(0,1)_{P^{-1}\mathbb{Z}} \times_p (0,1)_{P^{-1}\mathbb{Z}} := \{(r,s)\in(0,1)_{P^{-1}\mathbb{Z}}\times(0,1)_{P^{-1}\mathbb{Z}}:\ p(r)=p(s)\}$$

be the fiber product. Then,

$$\begin{aligned}
&\#\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P) \\
&\le \sum_{1\le u<v\le n}\ \sum_{\substack{x_u,x_v\in(0,1)_{P^{-1}\mathbb{Z}}\\ p(x_u)=p(x_v)}} \#\mathrm{J}(t,n-2;\mathbf{c}_{x_u,x_v};P) \\
&\ll_n \#\left((0,1)_{P^{-1}\mathbb{Z}}\times_p(0,1)_{P^{-1}\mathbb{Z}}\right)P^{n-2-t(t+1)/2} && \text{Theorem 5.2(ii)} \\
&\ll_p P^{n-1-t(t+1)/2}. && p\text{ is nonconstant}
\end{aligned} \tag{5.7}$$

Since $\mathcal{X}$ is in $\mathrm{J}(t;\mathcal{X},P)$, the tuple $\mathbf{x}$ associated to $\mathcal{X}$ is in the set $\mathrm{J}(t,n;\mathbf{c};P)$. Therefore, by Theorem 5.2(vi) and Eqs. (5.5) and (5.7),

$$\#\mathrm{J}_p(t;\mathcal{X};P) \asymp_n \#\mathrm{J}_p(t,n;\mathbf{c};P) = \#\mathrm{J}(t,n;\mathbf{c};P) - \#\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P) \gg_{t,n,p} P^{n-t(t+1)/2},$$

which proves the other direction of Eq. (5.2), and by Eqs. (5.6) and (5.7),

$$\#\widetilde{\mathrm{J}}_p(t;\mathcal{X};P) \le \#\widetilde{\mathrm{J}}_p(t,n;\mathbf{c};P) \ll_{t,\mathcal{X},p} P^{n-1-t(t+1)/2},$$

which proves Eq. (5.3). □

Corollary 5.4. *For every natural number t, there exists a rational t-design on $\mathcal{I}$.*

Proof. Clearly, $\mathcal{I}$ satisfies Condition 3.2, hence by Proposition 3.8, there exists some rational-weighted rational design $\mathcal{X}$ on $\mathcal{I}$ such that $\#\mathcal{X}\ge t$. Then, there exists an integer c such that $c\mathcal{X}$ is an integer-weighted design and $|c\mathcal{X}|\ge 3t^2 2^t - t$. Let $d:=1$ and let p be the identity map $\mathbb{Q}\to\mathbb{Q}$. Applying Theorem 5.3 to t, $\mathcal{X}$ and p, we know that for some positive integer P, $\mathrm{J}_p(t,\mathcal{X},P)$ is nonempty. Every element in $\mathrm{J}_p(t,\mathcal{X},P)$ is a rational t-design on $\mathcal{I}$. □

6. Algebraic path-connectivity

Algebraically path-connected space is a variant of ordinary path-connected where "algebraic paths" are used instead of ordinary paths.

Definition 6.1. Let $Z \subseteq \mathbb{R}^d$ be a subset and $\mathbb{F} \subseteq \mathbb{R}$ a subfield. The set Z is called $\mathbb{F}$-*algebraically path-connected* provided that for every finite subset $X \subseteq Z \cap \mathbb{F}^d$, there exists a polynomial map $p \in \mathbb{F}[x]^d$ such that $X \subseteq p((0,1)_{\mathbb{F}}) \subseteq Z \cap \mathbb{F}^d$.

In this section, we give a sufficient condition of $\mathbb{F}$-algebraic path-connectivity in Theorem 6.2 below.

Theorem 6.2. *Let $Z \subseteq \mathbb{R}^d$ be an open connected subset. For every subfield $\mathbb{F} \subseteq \mathbb{R}$, Z is $\mathbb{F}$-algebraically path-connected.*

In § 6.1, we use approximation theory to prove Theorem 6.2, which is used in the proof of Theorem 7.1 to construct a good path in an open connected subset of $\mathbb{R}^d$. In § 6.2, we show in Proposition 6.5 that the real unit sphere is not $\mathbb{F}$-algebraically path-connected for any subfield $\mathbb{F} \subseteq \mathbb{R}$. In Proposition 6.8, we show that the real unit sphere does not satisfy a property weaker than algebraic path-connectivity either. Note that if this weaker property were obtained for the real unit sphere, we could obtain from it the existence of rational spherical designs. Propositions 6.5 and 6.8 are not used in the remainder of the article.

6.1. **Algebraic path-connectivity of open connected sets.** For a uniformly continuous map f between metric spaces, let ω_f be the corresponding modulus of continuity, namely

$$\omega_f(\delta) := \sup_{\substack{x,y \in \text{domain } f \\ \text{dist}(x,y) \le \delta}} \text{dist}(f(x), f(y)).$$

A function f is *Dini-Lipschitz continuous* provided that $\omega_f(\delta) \log \delta$ converges to 0 as $\delta \to 0$. For $n \in N$, the n-th *Chebyshev polynomial of the first kind*, denoted by T_n, is the unique polynomial satisfying $T_n(\cos x) = \cos nx$. The roots of T_n,

$$\boldsymbol{\alpha}_n := \{\alpha_{n,k} : \ k \in [1,n]_{\mathbb{Z}}\}, \text{ where } \alpha_{n,k} := \cos \frac{2k-1}{2n}\pi, \tag{6.1}$$

are called *Chebyshev nodes*. The Lagrange interpolation of a function f at a finite set $\mathbf{n}$ of distinct nodes, denoted by $L(f, \mathbf{n})$, is given explicitly by

$$L(f,\mathbf{n})(x) := \sum_{\alpha \in \mathbf{n}} \left(f(\alpha) \prod_{\beta \in \mathbf{n} \setminus \{\alpha\}} \frac{x-\beta}{\alpha-\beta} \right).$$

Clearly, if $\mathbf{n} \subseteq \mathbb{F}$ and $f(\mathbf{n}) \subseteq \mathbb{F}$ for some field $\mathbb{F}$, then $L(f,\mathbf{n}) \in \mathbb{F}[x]$. Proposition 6.3 is a classical result on Chebyshev interpolation $L(f, \boldsymbol{\alpha}_n)$ of a real-valued function f on the interval $[-1,1]$.

Proposition 6.3 ([Riv69, Eq. 4.1.11, Theorem 4.5]). *Let f be a uniformly continuous real-valued function on the interval $[-1,1]$, and let ω_f be the* *modulus of continuity of f. For every positive integer n,*

$$\|f - L(f, \boldsymbol{\alpha}_{n+1})\|_\infty < 6\left(\frac{2}{\pi}\log n + 5\right)\omega_f(1/n).$$

In particular, if f is Dini-Lipschitz continuous, then $L(f, \boldsymbol{\alpha}_n)$ converges to f uniformly as $n \to +\infty$. □

Corollary 6.4 is a higher-dimensional version of Proposition 6.3.

Corollary 6.4. *Let $f : [-1, 1] \to \mathbb{R}^d$ be a Dini-Lipschitz continuous function. Then $L(f, \boldsymbol{\alpha}_n)$ converges to f uniformly as $n \to +\infty$.*

Proof. Suppose that $f = (f_1, \dots, f_d)$. All f_i's are Dini-Lipschitz continuous. The result follows by applying Proposition 6.3 to each f_i. □

Proof of Theorem 6.2. In this proof, a collection of indexed objects is denoted by the same letter in bold. For instance, the collection of all y_i's is denoted by $\mathbf{y}$.

Let $X \subseteq Z \cap \mathbb{F}^d$ be a finite subset. It suffices to construct a polynomial path $p : [-1, 1] \to \mathbb{R}^d$ given by polynomials with coefficients in $\mathbb{F}$, namely $p \in \mathbb{F}[x]^d$, such that $X \subseteq p((-1, 1)_{\mathbb{F}}) \subseteq Z$.

Since Z is both open and connected in the Euclidean space $\mathbb{R}^d$, for some natural number n, there exists a piecewise linear path in Z with $(n+1)$-pieces passing through all points in X at the boundary of pieces. Let $y_0 := -1$, $y_{n+1} := 1$ and $y_i := \alpha_{n,i}$ for $i \in [1, n]_{\mathbb{Z}}$ (see Eq. (6.1)). Then, there exists a piecewise linear function $\ell_{\mathbf{y}} : [-1, 1] \to \mathbb{R}^d$ with $n + 2$ nodes $\mathbf{y}$ satisfying $X \subseteq \{\ell_{\mathbf{y}}(y_i) : \ i \in [1, n]_{\mathbb{Z}}\}$. More precisely, $\ell_{\mathbf{y}}$ is a linear function on the interval $[y_i, y_{i+1}]$ for all $i \in [0, n]_{\mathbb{Z}}$.

Since $\operatorname{Im} \ell_{\mathbf{y}}$ is compact and Z is open, there exists an $\epsilon > 0$ such that for every $f : [-1, 1] \to \mathbb{R}^d$ satisfying $\|f - \ell_{\mathbf{y}}\|_\infty < \epsilon$, we have $\operatorname{Im} f \subseteq Z$.

The piecewise linear function $\ell_{\mathbf{y}}$ is Dini-Lipschitz continuous, hence by Corollary 6.4, for some sufficiently large odd natural number $m \in \mathbb{N}$,

$$\|\ell_{\mathbf{y}} - L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn})\|_\infty < \epsilon/2.$$

Let $v_j \in \mathbb{F}$ be a to-be-determined number near $\alpha_{mn,j}$ for $j \in [1, mn]_{\mathbb{Z}}$, $u_i := v_{im-(m-1)/2} \in \mathbb{F}$ for $i \in [1, n]_{\mathbb{Z}}$, and we further set $u_0 := y_0 = -1 \in \mathbb{F}$ and $u_{n+1} := y_{n+1} = 1 \in \mathbb{F}$. Then, there exists a piecewise linear function $\ell_{\mathbf{u}}$ with $n + 2$ nodes $\mathbf{u}$ such that $\ell_{\mathbf{u}}(u_i) = \ell_{\mathbf{y}}(y_i)$ for $i \in [0, n + 1]_{\mathbb{Z}}$. It is clear that $\operatorname{Im} \ell_{\mathbf{u}} = \operatorname{Im} \ell_{\mathbf{y}}$. Now, for each $j \in [1, mn]_{\mathbb{Z}}$, let v_j be sufficiently close to $\alpha_{mn,j}$. For every $i \in [1, n]_{\mathbb{Z}}$, since $\alpha_{n,i} = \alpha_{mn,im-(m-1)/2}$, u_i is sufficiently close to $y_i = \alpha_{n,i}$ as well. Therefore, we choose a suitable $\mathbf{v}$ such that

$$\|L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn}) - L(\ell_{\mathbf{u}}, \mathbf{v})\|_\infty < \epsilon/2.$$

By subadditivity of norms,

$$\|\ell_{\mathbf{y}} - L(\ell_{\mathbf{u}}, \mathbf{v})\|_\infty \le \|\ell_{\mathbf{y}} - L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn})\|_\infty + \|L(\ell_{\mathbf{y}}, \boldsymbol{\alpha}_{mn}) - L(\ell_{\mathbf{u}}, \mathbf{v})\|_\infty < \epsilon,$$

hence $\operatorname{Im} L(\ell_{\mathbf{u}}, \mathbf{v}) \subseteq Z$.

Note that, for $i \in [1,n]_{\mathbb{Z}}$, since $u_i = v_{im-(m+1)/2}$,

$$L(\ell_{\mathbf{u}},\mathbf{v})(u_i) = L(\ell_{\mathbf{u}},\mathbf{v})(v_{im-(m+1)/2}) = \ell_{\mathbf{u}}(v_{im-(m+1)/2}) = \ell_{\mathbf{u}}(u_i) = \ell_{\mathbf{y}}(y_i),$$

which implies that $X \subseteq \{L(\ell_{\mathbf{u}},\mathbf{v})(u_i) : \ i \in [1,n]_{\mathbb{Z}}\} \subseteq L(\ell_{\mathbf{u}},\mathbf{v})((-1,1)_{\mathbb{F}})$. Moreover, elements in $\mathbf{v}$ are in $\mathbb{F}$, points in X have coordinates in $\mathbb{F}$ and $\ell_{\mathbf{u}}$ is piecewise $\mathbb{F}$-linear, so we have $L(\ell_{\mathbf{u}},\mathbf{v}) \in \mathbb{F}[x]^d$. Thus, $L(\ell_{\mathbf{u}},\mathbf{v})$ is a desired path. □

6.2. **Non-algebraic path-connectivity of the real spheres.**

Proposition 6.5. *Let $d \in \mathbb{N}$. There do not exist non-constant polynomial paths on $\mathcal{S}^d$. In particular, the real sphere $\mathcal{S}^d$ is not $\mathbb{F}$-algebraically path-connected for any subfield $\mathbb{F} \subseteq \mathbb{R}$.*

Proof. Suppose that there exists a polynomial map $p = (p_0,\dots,p_d) \in \mathbb{R}[x]^d$ such that $p(I) \subseteq \mathcal{S}^d$ for some nontrivial interval I. Since the leading coefficient of each p_i^2 is positive and $\sum_{i=0}^d p_i^2 = 1$, every p_i is constant, which means that p is constant. □

Let $\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the algebra of integer-valued polynomials. Let Δ be the standard forward difference operator, namely $(\Delta f)(x) := f(x+1) - f(x)$ for a function f, and D the differential operator, namely $\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}\,x}$. Next, we present an elementary proof of a result on integer-valued polynomials. Note that this result can be essentially deduced from a stronger result in [DLS64], which is proved using algebraic number theory.

Theorem 6.6. *Let $f, g \in \mathrm{Int}(\mathbb{Z})$. If $f(\mathbb{N}) \subseteq g(\mathbb{N})$, then $f = g \circ h$ for some $h \in \mathrm{Int}(\mathbb{Z})$. Moreover, the polynomial h is unique unless either f or g is a constant.*

Proof. The result holds trivially when either f or g is a constant. From now on, we assume that f and g are not constants. The polynomials f and g are strictly monotone on a neighborhood of $+\infty$. So, for sufficiently large $x_0 \in \mathbb{N}$, the function $\eta := g^{-1} \circ f : \mathbb{R}_{\geq x_0} \to \mathbb{R}_{\geq 0}$ is well-defined, which has the order $\eta(x) \asymp x^{\frac{\deg f}{\deg g}}$ as $x \to +\infty$.

Let k be a natural number. We first compute the k-th derivative of η. For a rational function $r \in \mathbb{Q}(x)$ and a polynomial $p \in \mathbb{Q}[x]$,

$$\mathrm{D}((r\circ\eta)\cdot p) = \left(\frac{\mathrm{D}r}{\mathrm{D}g} \circ \eta\right) \cdot (\mathrm{D}f) \cdot p + (r\circ\eta)\cdot(\mathrm{D}p).$$

Analyzing the order of both sides the equation above, we have $(\mathrm{D}((r\circ\eta)\cdot p))(x) \ll x^{-1}((r\circ\eta)\cdot p)(x)$ as $x \to +\infty$. By induction, it is easy to show that $(\mathrm{D}^k\eta)(x) = \big(\mathrm{D}^k((x\circ\eta)\cdot 1)\big) \ll x^{-k}\eta(x)$ as $x \to +\infty$. We now compute the k-th difference of η using the well-known relation

$$(\Delta^k\eta)(x) = (\mathrm{D}^k\eta)(x+\theta_x),$$

for some $\theta_x \in [0, k]$. For $k := \left\lceil \frac{\deg f}{\deg g} \right\rceil + 1$,

$$(\Delta^k \eta)(x) \asymp (\mathrm{D}^k \eta)(x) \ll x^{-k} \eta(x) \ll x^{-1},$$

as $x \to +\infty$. The condition $f(\mathbb{N}) \subseteq g(\mathbb{N})$ forces $\Delta^k \eta$ to take integer values on $\mathbb{Z}_{\geq x_0}$. So, there exists an $x_1 \in \mathbb{N}$ such that $(\Delta^k \eta)(n) = 0$ for all $n \in \mathbb{Z}_{\geq x_1}$, as a result, there exists a unique $h \in \mathrm{Int}(\mathbb{Z})$ such that η and h agree on $\mathbb{Z}_{\geq x_1}$. Thus, the polynomials $f = g \circ \eta$ and $g \circ h$ agree on $\mathbb{Z}_{\geq x_1}$, which implies that $f = g \circ h$. □

Corollary 6.7. *Let $f : I \to \mathbb{R}$ be a continuous function on a real interval I such that $f(I \cap \mathbb{Q}) \subseteq \mathbb{Q}$ and f^2 is a polynomial function. Then, f is a piecewise polynomial function.*

Proof. The result holds trivially when f is a constant or I is trivial. Now, we only need to consider nonconstant f and nontrivial I. By a linear change of variable with coefficients in $\mathbb{Q}$, without loss of generality, we assume that $[0, 1] \subseteq I$.

Let $q \in \mathbb{R}[x]$ be the polynomial such that $q = f^2$ on I. Since q maps rational numbers to rational numbers, $q \in \mathbb{Q}[x]$. Let c be a nonzero common multiple of the denominators of the coefficients of the polynomial q, and let $r := c^2(x+1)^{2 \deg q} q(\frac{1}{x+1}) \in \mathbb{Z}[x]$. Since $r = \left(c(x+1)^{\deg q} f(\frac{1}{x+1})\right)^2$ and $f(\frac{1}{m+1}) \in \mathbb{Q}$ for all $m \in \mathbb{N}$, we have $r(\mathbb{N}) \subseteq \{n^2 : n \in \mathbb{N}\}$. By Theorem 6.6, there exists a polynomial $h \in \mathbb{Q}[x]$ such that $h^2 = r$. So there exists a polynomial $p \in \mathbb{Q}[x]$ such that $q = p^2$, hence $|f| = |p|$ on I. Therefore, by continuity of f, f is a piecewise polynomial function. □

Proposition 6.8. *There do not exist non-constant continuous paths $p = (p_0, \ldots, p_d) : [0, 1] \to \mathcal{S}^d$ such that for each i, p_i^2 is a polynomial function, and p maps rational points to rational points.*

Proof. Suppose that there exists a desired path p. By Corollary 6.7, all p_i's are piecewise polynomial functions. In other words, p is a piecewise polynomial path. Thus, on each piece of p, p is a constant path by Proposition 6.5. By the continuity of p, p is a constant path on $[0, 1]$, which contradicts our assumption. □

7. Proofs of the main results

In § 7.1, we present a technical result, Theorem 7.1, on the existence of designs on algebraically path-connected spaces. Then in § 7.2, we derive all the main results mentioned in § 1 from Theorem 7.1. Note that we use Definition 3.1 as the definition of designs, unless explicitly stated otherwise.

7.1. Rational designs on algebraically path-connected spaces.

Theorem 7.1. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ with $d \geq 1$ be a levelling space satisfying Condition 3.2, and let t be a natural number. Denote by $\mathrm{J}(t, n; \mathcal{Z}; P)$ the set of* *all rational t-designs $\mathcal{X}$ on $\mathcal{Z}$ of size n satisfying $\mathcal{X} \subseteq \mathcal{Z} \cap (P^{-1}\mathbb{Z})^d$. If $\mathcal{Z}$ is $\mathbb{Q}$-algebraically path-connected, then there exist positive integers d_0 and n_0 such that for every natural number $n \geq n_0$,*

$$\#\mathrm{J}(t, n; \mathcal{Z}; P) \gg_{t,n,\mathcal{Z}} P^{(n - td_0(td_0+1)/2)/d_0},$$

as $P \to \infty$ in the domain $P_0\,\mathbb{Z}_{>0}$ for some positive integer P_0.

The strategy for proving Theorem 7.1 is as follows: In Step 1, we apply the algebraic path-connectivity of $\mathcal{Z}$ to find a suitable path $\mathcal{I} \to \mathcal{Z}$. In Step 2, we use this path to construct a certain weighted levelling space in $\mathcal{I} \cap \mathbb{Q}$. In Step 3, we convert the measure of this levelling space to a measure taking integer values. In Step 4, we construct an equivalent integer-weighted levelling space in $\mathcal{I} \cap \mathbb{Q}$ for each sufficiently large total measure. In Step 5, we regard each integer-weighted levelling space as a multiset, and separate the repeated elements to obtain levelling spaces in $\mathcal{I} \cap \mathbb{Q}$ equipped with the counting measures. In Step 6, we lift these levelling spaces to levelling spaces in $\mathcal{Z} \cap \mathbb{Q}^d$, which will be rational designs on $\mathcal{Z}$. At the end, in Step 7, we analyze the number of rational designs on $\mathcal{Z}$.

Proof. We are going to construct several levelling spaces $\mathcal{X}, \mathcal{X}', \mathcal{X}_n \subseteq \mathcal{Z} \cap \mathbb{Q}^d$ and $\mathcal{Y}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{Y}}_n, \mathcal{Y}_n \subseteq \mathcal{I} \cap \mathbb{Q}$, and $\mathcal{X}_n$ will be a rational t-design on $\mathcal{Z}$ of size n. In the following diagram, we summarize the sequence of constructions.

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\text{Lemma 3.4}"] & \mathcal{X} \arrow[r, "\text{Lemma 2.16}"] & \mathcal{X}' \arrow[d, bend left, "\text{Lemma 2.6(v)}"] & & & \mathcal{X}_n \\
\mathcal{I} \arrow[u, "p"] & & \mathcal{Y} \arrow[u, bend left, "p"] \arrow[r, "\text{Lemma 3.6}"'] & \widetilde{\mathcal{Y}} \arrow[r, "\text{Theorem 4.5}"'] & \widetilde{\mathcal{Y}}_n \arrow[r, "\text{Theorem 5.3}"'] & \mathcal{Y}_n \arrow[u, "p"]
\end{tikzcd}$$

Step 1. Since $\mathcal{Z}$ satisfies Condition 3.2, according to Lemma 3.4, there exists a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate (see Definition 2.12) weighted rational t-design $\mathcal{X}$ on $\mathcal{Z}$ with support X. By the $\mathbb{Q}$-algebraic path-connectivity of $\mathcal{Z}$ (see Definition 6.1), there exists a polynomial path $p \in \mathbb{Q}[x]^d$ such that $X \subseteq p(\mathcal{I} \cap \mathbb{Q}) \subseteq \mathcal{Z} \cap \mathbb{Q}^d$. Let $t_0 := t \deg p$.
Step 2. Let S be a finite $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate subset of $\mathcal{I} \cap \mathbb{Q}$ such that $\#S \geq t_0$. Since X is the support of $\mathcal{X}$, which is a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design on $\mathcal{Z}$, according to Lemma 2.16, $X' := X \cup p(S)$ is the support of a finite $\mathcal{P}^t[\mathcal{Z}]$-nondegenerate weighted rational t-design $\mathcal{X}'$ on $\mathcal{Z}$. Let $Y := p^{-1}(X') \cap \mathcal{I} \cap \mathbb{Q}$. The map p then induces a surjection $p|_Y : Y \twoheadrightarrow X'$, so by Lemma 2.6(v), Y is the support of a levelling space $\mathcal{Y} \subseteq \mathcal{I} \cap \mathbb{Q}$ such that $p(\mathcal{Y}) = \mathcal{X}'$. Moreover, $S \subseteq Y$, hence $\#Y \geq \#S \geq t_0$ and $\mathcal{Y}$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate because S is $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate.
Step 3. Since $\mathcal{Y}$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-nondegenerate, by Lemma 3.6, there exists an integer-weighted (see Definition 2.2) levelling space $\widetilde{\mathcal{Y}} \subseteq \mathcal{I} \cap \mathbb{Q}$ such that its support coincides with the support of $\mathcal{Y}$ and it is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent (see Definition 2.8) to $\mathcal{Y}$.

Step 4. By Theorem 4.5, there exists a natural number n_0, which we assume to be at least $3t_0^2 2_0^t - t_0$, such that for every $n \geq n_0$, there exists an integer-weighted levelling space $\widetilde{\mathcal{Y}}_n \subseteq \mathcal{I} \cap \mathbb{Q}$ such that its total measure is n and it is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent to $\widetilde{\mathcal{Y}}$.
Step 5. Let $\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)$ be the set of levelling spaces $\mathcal{Y}_n \subseteq \mathcal{I} \cap Q^{-1}\mathbb{Z}$ such that $\mathcal{Y}_n$ carries the counting measure with total measure n, p is injective on $\widetilde{\mathcal{Y}}_n$, and $\mathcal{Y}_n$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent to $\widetilde{\mathcal{Y}}_n$. By Theorem 5.3,

$$\#\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q) \asymp_{t_0, \widetilde{\mathcal{Y}}_n, p} Q^{n - t_0(t_0+1)/2}$$

as $Q \to \infty$ in the domain $Q_0\mathbb{Z}_{>0}$ for some positive integer Q_0 depending on t_0 and $\widetilde{\mathcal{Y}}_n$.
Step 6. Let $P := Q^{\deg p}$. We will show that p induces an inclusion

$$\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q) \hookrightarrow \mathrm{J}(t, n; \mathcal{Z}; P).$$

Let $\mathcal{Y}_n$ be an arbitrary element in $\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)$ for some $Q \in Q_0\mathbb{Z}_{>0}$. By the transitivity of $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalence, $\mathcal{Y}_n$ is $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent to $\mathcal{Y}$. Let $\mathcal{X}_n := p(\mathcal{Y}_n) \subseteq \mathcal{Z} \cap (P^{-1}\mathbb{Z})^d$. By the injectivity of p on $\mathcal{Y}_n$, $\mathcal{X}_n$ has the counting measure with total measure n. Since $\mathcal{Y}_n$ and $\mathcal{Y}$ are $\mathcal{P}^{t_0}[\mathcal{I}]$-equivalent and $p^*\mathcal{P}^t[\mathcal{Z}] \subseteq \mathcal{P}^{t_0}[\mathcal{I}]$, they are also $p^*\mathcal{P}^t[\mathcal{Z}]$-equivalent, where $p^*\mathcal{P}^t[\mathcal{Z}] = \{f \circ p : f \in \mathcal{P}^t[\mathcal{Z}]\}$ is the pullback of polynomials in $\mathcal{P}^t[\mathcal{Z}]$. Therefore, by Lemma 2.10, $\mathcal{X}_n = p(\mathcal{Y}_n)$ and $\mathcal{X}' = p(\mathcal{Y})$ are $\mathcal{P}^t[\mathcal{Z}]$-equivalent. Furthermore, since the levelling space $\mathcal{X}'$ is a weighted t-design on $\mathcal{Z}$, $\mathcal{X}_n$ is a rational t-design on $\mathcal{Z}$ of size n. So, p induces an inclusion that maps $\mathcal{Y}_n \in \mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q)$ to $\mathcal{X}_n \in \mathrm{J}(t, n; \mathcal{Z}; P)$.
Step 7. Set $d_0 := \deg p$ and $P_0 := Q_0^{d_0}$. We employ the inclusion obtained from Step 6 and the asymptotic formula in Step 5, and find

$$\#\mathrm{J}(t, n; \mathcal{Z}; P) \geq \#\mathrm{J}_p(t_0; \widetilde{\mathcal{Y}}_n; Q) \asymp_{t_0, \widetilde{\mathcal{Y}}_n, p} Q^{n - t_0(t_0+1)/2} = P^{(n - td_0(td_0+1)/2)/d_0}$$

as $P \to \infty$ in the domain $P_0\mathbb{Z}_{>0}$, where $d_0 := \deg p$, t_0 and p are determined by $t, n, \mathcal{Z}$ in Step 1, and $\widetilde{\mathcal{Y}}_n$ is determined by $t, n, \mathcal{Z}$ in Step 4. □

7.2. Rational designs on open connected spaces.

Proof of Theorem 1.4. Since $\mathcal{Z}$ is open connected, it is also $\mathbb{Q}$-algebraically path-connected by Theorem 6.2. Then, by Theorem 7.1, there exists an n_0 such that for all natural number $n \geq n_0$, the set $\mathrm{J}(t, n; \mathcal{Z}; P)$ is nonempty for some P, hence there exists a rational t-design $\mathcal{X}$ on $\mathcal{Z}$ of size n. □

Corollary 7.2. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a levelling space satisfying Condition 3.2 and such that the measure of every point in $\mathcal{Z}$ is 0. Let t be a natural number. If $\mathcal{Z}$ is open connected, then there exist infinitely many disjoint rational t-designs on $\mathcal{Z}$ of size n for all sufficiently large n.*

Proof. Let m be an arbitrary natural number, and suppose that $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are m rational t-designs on $\mathcal{Z}$ of size n. Let $\mathcal{Z}' := \mathcal{Z} \setminus \bigcup_{i=1}^m \mathcal{X}_i$ be equipped with the measure induced from $\mathcal{Z}$. Since $\mathcal{Z}$ is open connected and $\mathcal{X}_i$'s are finite, clearly $\mathcal{Z}'$ is open connected. Therefore, by Theorem 1.4, there exists a rational t-design $\mathcal{X}_{m+1}$ on $\mathcal{Z}'$ of size n. Since the measure of $\mathcal{Z} \setminus \mathcal{Z}'$ is zero, $\mathcal{X}_{m+1}$ is a rational t-design on $\mathcal{Z}$ of size n as well. Furthermore, the $m+1$ rational t-designs $\mathcal{X}_1, \ldots, \mathcal{X}_{m+1}$ are disjoint. The result then follows from an induction on m. $\square$

Proof of Theorem 1.3. Consider the $(d-1)$-sphere $\mathcal{S}^{d-1}$ in $\mathcal{S}^d$ given by $\{(x_0, \ldots, x_d) \in \mathcal{S}^d : \ x_0 = 0\}$. It is closed and has measure 0 in $\mathcal{S}^d$. So, we get a levelling space $\mathcal{S}^d \setminus \mathcal{S}^{d-1}$ equipped with the measure induced from $\mathcal{S}^d$.

Let $p : \ \mathcal{S}^d \to \mathbb{R}^d$ be the projection map given by $(x_0, \ldots, x_d) \mapsto (x_1, \ldots, x_d)$, and let $\mathcal{B}^d$ be the image of $\mathcal{S}^d \setminus \mathcal{S}^{d-1}$ under p. As a topological space, $\mathcal{B}^d$ is a d-dimensional real open unit ball, and it is equipped with the pushforward measure (see Definition 2.5). By applying Theorem 1.4 to $\mathcal{B}^d$, there exists a rational t-design $\mathcal{X}$ on $\mathcal{B}^d$. Let $\mathcal{Y} := p^{-1}(\mathcal{X})$ be defined as in Lemma 2.6(v). For each point in $\mathcal{Y}$, all its coordinates are rational except possibly the first coordinate. By Lemma 2.6(vi), $2\,\mathcal{Y}$ is equipped with the counting measure. We claim that $2\,\mathcal{Y}$ is a spherical t-design on $\mathcal{S}^d$, hence the result.

It suffices to show that for every monic monomial $f \in \mathbb{R}[\mathcal{S}^d]$, Eq. (1.1) holds for $2\,\mathcal{Y}$. Indeed, the vector space $\mathbb{R}[\mathcal{S}^d]$ has a decomposition $\mathbb{R}[\mathcal{B}^d] \oplus x_0\,\mathbb{R}[\mathcal{B}^d]$. For each $f \in x_0\,\mathbb{R}[\mathcal{B}^d]$, it is easy to see by symmetry that both sides of Eq. (1.1) vanish, and for each $f \in \mathbb{R}[\mathcal{B}^d]$, Eq. (1.1) follows from the fact that $\mathcal{X}$ is a t-design on $\mathcal{B}^d$. Therefore, $2\,\mathcal{Y}$ is a spherical t-design. $\square$

7.3. **Rational designs on rational simplicial complexes.** Recall from § 2.3 that $\operatorname{aff} S$, the affine space generated by S, is the set of all affine linear combinations of S, and $\operatorname{reint} S$, the relative interior of S, is the set of all positive convex combinations of S.

Lemma 7.3. *Let $\mathcal{Z} \subseteq \mathbb{R}^d$ be a convex polytope equipped with the k-dimensional Hausdorff measure where k is the dimension of $\mathcal{Z}$. Suppose that $k \geq 1$ and the vertices of $\mathcal{Z}$ are rational points. Let t be a natural number. Then, there are infinitely many rational t-designs on the relative interior* $\operatorname{reint} \mathcal{Z}$ *of $\mathcal{Z}$ of size n for all sufficiently large n.*

Proof. Let $\operatorname{aff} \mathcal{Z}$ be the affine space generated by $\mathcal{Z}$. The affine space $\operatorname{aff} \mathcal{Z}$ can be identified with the Euclidean space $\mathbb{R}^k$ such that rational points $\operatorname{aff} \mathcal{Z} \cap \mathbb{Q}^d$ in $\operatorname{aff} \mathcal{Z}$ and rational points $\mathbb{Q}^k$ are identified.

In $\operatorname{aff} \mathcal{Z}$, the convex polytope $\mathcal{Z}$ is of dimension k, and the k-dimensional Hausdorff measure is just the Lebesgue measure in $\mathbb{R}^k$ up to a positive scalar. Consider the relative interior $\operatorname{reint} \mathcal{Z}$ of $\mathcal{Z}$, which is also the interior of $\mathcal{Z}$ in $\operatorname{aff} \mathcal{Z}$. Since $k \geq 1$, $\operatorname{reint} \mathcal{Z}$ is open and connected in $\operatorname{aff} \mathcal{Z}$, hence Condition 3.2(i) holds. Since the vertices of $\mathcal{Z}$ are rational points, Condition 3.2(ii) is also satisfied. We apply Corollary 7.2 to $\operatorname{reint} \mathcal{Z}$, which is viewed as a subset of the Euclidean space $\operatorname{aff} \mathcal{Z}$, and conclude that there are infinitely many disjoint rational t-designs on $\operatorname{reint} \mathcal{Z}$. $\square$

Proof of Theorem 1.5. The result holds trivially when $k = 0$, and we assume $k \geq 1$ from now on. Let $\mathcal{K}_1, \dots, \mathcal{K}_m$ be all the k-simplexes in the simplicial k-complex $\mathcal{Z}$, equipped with the k-dimensional Hausdorff measure. Since every $\mathcal{K}_i$ is a k-simplex with rational vertices, the total measure $|\mathcal{K}_i|$ is a rational number. Without loss of generality, we assume that $|\mathcal{K}_i|$ is a sufficiently large integer for all $i \in [1, m]_{\mathbb{Z}}$ (otherwise we can replace $\mathcal{Z}$ with its scalar $c\,\mathcal{Z}$ for some large suitable positive integer c).

For each $i \in [1, m]_{\mathbb{Z}}$, we apply Lemma 7.3 to $\mathcal{K}_i$ and obtain a rational t-design $\mathcal{X}_i$ on $\operatorname{reint} \mathcal{K}_i$ of size $|\mathcal{K}_i|$. We set $\mathcal{X} := \mathcal{X}_1 + \cdots + \mathcal{X}_n$. For all polynomials $f \in \mathcal{P}^t[\mathcal{Z}]$,

$$
\begin{aligned}
&\frac{1}{|\,\mathcal{Z}\,|} \int_Z f \, \mathrm{d}\, \mu_Z \\
=&\frac{1}{\sum_{i=1}^m |\mathcal{K}_i|} \sum_{i=1}^m |\mathcal{K}_i| \frac{1}{|\operatorname{reint} \mathcal{K}_i|} \int_{\operatorname{reint} K_i} f \, \mathrm{d}\, \mu_{K_i} && \bigcup_{i=1}^m \operatorname{reint} K_i \hookrightarrow Z, |\mathcal{K}_i| = |\operatorname{reint} \mathcal{K}_i| \\
=&\frac{1}{\sum_{i=1}^m |\mathcal{K}_i|} \sum_{i=1}^m |\mathcal{K}_i| \frac{1}{|\,\mathcal{X}_i\,|} \int_{X_i} f \, \mathrm{d}\, \mu_{X_i} && \mathcal{X}_i \text{ is a } t\text{-design on } \mathcal{K}_i \\
=&\frac{1}{\sum_{i=1}^m |\,\mathcal{X}_i\,|} \sum_{i=1}^m \int_{X_i} f \, \mathrm{d}\, \mu_{X_i} && |\,\mathcal{X}_i\,| = |\mathcal{K}_i| \\
=&\frac{1}{|\,\mathcal{X}\,|} \int_X f \, \mathrm{d}\, \mu_X, && \mathcal{X} := \mathcal{X}_1 + \cdots + \mathcal{X}_n
\end{aligned}
$$

which proves that $\mathcal{X}$ is a rational t-design on $\mathcal{Z}$. □

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