ON q-SCHUR ALGEBRAS CORRESPONDING TO HECKE ALGEBRAS OF TYPE B

CHUN-JU LAI, DANIEL K. NAKANO, AND ZIQING XIANG

ABSTRACT. In this paper the authors investigate the q-Schur algebras of type B that were constructed earlier using coideal subalgebras for the quantum group of type A. The authors present a coordinate algebra type construction that allows us to realize these q-Schur algebras as the duals of the dth graded components of certain graded coalgebras. Under suitable conditions a Morita equivalence theorem is proved that demonstrates that the representation theory reduces to the q-Schur algebra of type A. This enables the authors to address the questions of cellularity, quasi-hereditariness and representation type of these algebras. Later it is shown that these algebras realize the 1-faithful quasi hereditary covers of the Hecke algebras of type B.

1. INTRODUCTION

1.1. Schur-Weyl duality has played a prominent role in the representation theory of groups and algebras. The duality first appeared as method to connect the representation theory of the general linear group GL_n and the symmetric group Σ_d. This duality carries over naturally to the quantum setting by connecting the representation theory of quantum GL_n and the Hecke algebra $\mathcal{H}_q(\Sigma_d)$ of the symmetric group Σ_d.

Let $U_q(\mathfrak{gl}_n)$ be the Drinfeld-Jimbo quantum group. Jimbo showed in [Jim86] that there is a Schur duality between $U_q(\mathfrak{gl}_n)$ and $\mathcal{H}_q(\Sigma_d)$ on the d-fold tensor space of the natural representation V of $U_q(\mathfrak{gl}_n)$. The q-Schur algebra of type A, $S_q^{\mathrm{A}}(n,d)$, is the centralizer algebra of the $\mathcal{H}_q(\Sigma_d)$-action on $V^{\otimes d}$.

It is well-known that the representation theory for $U_q(\mathfrak{gl}_n)$ is closely related to the representation theory for the quantum linear group GL_n. The polynomial representations GL_n coincide with modules of $S_q^{\mathrm{A}}(n,d)$ with $d>0$. The relationship between objects are depicted as below:

$$\begin{array}{ccccccc} K[M_q^{\mathrm{A}}(n)]^* & \hookleftarrow & U_q(\mathfrak{gl}_n) & & & & \\ \downarrow & & \downarrow & & & & \\ K[M_q^{\mathrm{A}}(n)]_d^* & \simeq & S_q^{\mathrm{A}}(n,d) & \rightharpoonup & V^{\otimes d} & \leftharpoonup & \mathcal{H}_q(\Sigma_d) \end{array}$$

The algebra $U_q(\mathfrak{gl}_n)$ embeds in the dual of the quantum coordinate algebra $K[M_q^{\mathrm{A}}]$; while $S_q^{\mathrm{A}}(n,d)$ can be realized as its d-th degree component. The reader is referred to [PW91] for a thorough treatment of the subject.

The Schur algebra $S_q^{\mathrm{A}}(n,d)$ and the Hecke algebra $\mathcal{H}_q(\Sigma_d)$ are structurally related when $n \geqslant d$.

- There exists an idempotent $e \in S_q^{\mathrm{A}}(n,d)$ such that $eS_q^{\mathrm{A}}(n,d)e \simeq \mathcal{H}_q(\Sigma_d)$;
- An idempotent yields the existence of Schur functor $\mathrm{Mod}(S_q^{\mathrm{A}}(n,d)) \to \mathrm{Mod}(\mathcal{H}_q(\Sigma_d))$;
- $S_q^{\mathrm{A}}(n,d)$ is a (1-faithful) quasi-hereditary cover of $\mathcal{H}_q(\Sigma_d)$[1]

1.2. Our paper aims to investigate the representation theory of the q-Schur algebras of type B that arises from the coideal subalgebras for the quantum group of type A. We construct, for type B = C,

Research of the second author was supported in part by NSF grant DMS-1701768.

[1]The algebra $S_q^{\mathrm{A}}(n,d)$ is 1-faithful under the conditions that q is not a root of unity or if q^2 is a primitive ℓth root of unity then $\ell \geqslant 4$.

the following objects in the sense that all favorable properties mentioned in the previous section hold:

$$\begin{array}{ccccccc} K[M^{\mathrm{B}}_{Q,q}(n)]^* & \leftrightarrow & U^{\mathrm{B}}_{Q,q}(n) & & & & \\ \downarrow & & \downarrow & & & & \\ K[M^{\mathrm{B}}_{Q,q}(n)]^*_d & \simeq & S^{\mathrm{B}}_{Q,q}(n,d) & \rightharpoondown & V_{\mathrm{B}}^{\otimes d} & \leftharpoondown & \mathcal{H}^{\mathrm{B}}_{Q,q}(d) \end{array}$$

For our purposes it will be advantageous to work in more general setting with two parameters q and Q, and construct the analogs $K[M^{\mathrm{B}}_{Q,q}(n)]$ of the quantum coordinate algebras. Then we prove that the dth degree component of $K[M^{\mathrm{B}}_{Q,q}(n)]^*$ is isomorphic to the type B q-Schur algebras. The coordinate approach provide tools to study the representation theory for the algebra $K[M^{\mathrm{B}}_{Q,q}(n)]^*$ and for the q-Schur algebras simultaneously. The algebra $U^{\mathrm{B}}_{Q,q}(n)$, unlike $U_q(\mathfrak{gl}_n)$, does not have an obvious comultiplication. Therefore, its dual object, $K[M^{\mathrm{B}}_{Q,q}(n)]$, should be constructed as a coalgebra; while in the earlier situation $K[M^{\mathrm{A}}_q(n)]$ is a bialgebra.

In the second part of the paper an isomorphism theorem between the q-Schur algebras of type B and type A (under an invertibility condition) is established:

$$S^{\mathrm{B}}_{Q,q}(n,d) \cong \begin{cases} \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(r,i) \otimes S^{\mathrm{A}}_q(r,d-i) & \text{if } n = 2r; \\ \bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(r+1,i) \otimes S^{\mathrm{A}}_q(r,d-i) & \text{if } n = 2r+1. \end{cases} \tag{1.2.1}$$

One can view this as a "lifting" of the Morita equivalence (via the Schur functor)

$$\mathcal{H}^{\mathrm{B}}_{Q,q}(d) \underset{\mathrm{Mor}}{\simeq} \prod_{i=0}^{d} \mathcal{H}_q(\Sigma_i) \otimes \mathcal{H}_q(\Sigma_{d-i}), \tag{1.2.2}$$

between Hecke algebras proved by Dipper-James [DJ92]:

As a corollary of our isomorphism theorem, we obtain favorable properties for our coideal Schur algebras, see Section 5–8. In particular, with the Morita equivalance we are able to show that $S^{\mathrm{B}}_{Q,q}(n,d)$ is a cellular algebra and quasi-hereditary. Moreover, in Section 7, we are able give a complete classification of the representation type of $S^{\mathrm{B}}_{Q,q}(n,d)$. In the following section (Section 8), we are able to demonstrate that under suitable conditions, the Schur algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is a quasi-hereditary one-cover for $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$. We also exhibit how the representation theory of $S^{\mathrm{B}}_{Q,q}(n,d)$ is related to Rouquier's Schur-type algebras that arise from the category $\mathcal{O}$ for rational Cherednik algebras.

In the one-parameter case (i.e., $q = Q$), the algebra $U^{\mathrm{B}}_q(n)$ is the coideal subalgebra $\mathbf{U}^\imath$ or $\mathbf{U}^\jmath$ of $U_q(\mathfrak{gl}_n)$ in [BW13] (see also [ES18]). To our knowledge, there is no general theory for finite-dimensional representations for the coideal subalgebras (cf. [Le17] for establishing their Cartan subalgebras), and in some way our paper aims to establish results about "polynomial" representations for $U^{\mathrm{B}}_q(n)$. The corresponding Schur algebras therein are denoted by $\mathbf{S}^\imath$ or $\mathbf{S}^\jmath$ to emphasize the fact that they arise from certain quantum symmetric pairs of type A III/IV associated with involutions $\imath$ or $\jmath$ on a Dynkin diagram of type A_n. Namely, we have the identification below:

$$U^{\mathrm{B}}_q(n) \equiv \begin{cases} \mathbf{U}^\jmath_r & \text{if } n = 2r+1; \\ \mathbf{U}^\imath_r & \text{if } n = 2r, \end{cases} \qquad S^{\mathrm{B}}_q(n,d) \equiv \begin{cases} \mathbf{S}^\jmath(r,d) & \text{if } n = 2r+1; \\ \mathbf{S}^\imath(r,d) & \text{if } n = 2r. \end{cases}$$

There are many cases when the Morita equivalence will hold, in particular when (i) q is generic, (ii) q is an odd root of unity, or (iii) q is an (even) ℓth root of unity if $\ell > 4d$.

There are several generalizations of the q-Schur duality for type B. A comparison of the algebras regarding the aforementioned favorable properties will be given in Section 9. Since all these algebras are the centralizing partners of certain Hecke algebra actions, they are different from the ones appearing in the Schur duality (see [H11]) for type B/C quantum groups, and are different from the coordinate algebras studied by Doty [Do98].

Acknowledgements. We thank Huanchen Bao, Stefan Kolb, and Weiqiang Wang for useful discussions. The first author thanks the Academia Sinica for the support and hospitality during the completion of this project.

2. Quantum coordinate (co)algebras

2.1. **Quantum matrix spaces.** Let K be a field containing elements q, Q. Denote the (quantum) commutators by

$$[A,B]_x = AB - xBA \quad (x \in K), \quad [A,B] = [A,B]_1. \tag{2.1.1}$$

We define the (type A) quantum matrix spaces following [PW91, §3.5] but with a shift on the index set as below:

$$I(n) = \begin{cases} [-r,r] \cap \mathbb{Z} & \text{if } n = 2r+1; \\ [-r,r] \cap \mathbb{Z} - \{0\} & \text{if } n = 2r. \end{cases} \tag{2.1.2}$$

Let $M_q^{\mathrm{A}} = M_q^{\mathrm{A}}(n)$ be the quantum analog of the space of $n \times n$ matrices indexed by $I(n)$, and let $K[M_q^{\mathrm{A}}] = K[x_{ij}; i,j \in I(n)]/J_q^{\mathrm{A}}(n)$ be the associative K-algebra where $J_q^{\mathrm{A}}(n)$ is the two-sided ideal of $K[x_{ij}]$ generated by

$$[x_{ki}, x_{kj}]_{q^{-1}}, \qquad i > j \tag{2.1.3}$$

$$[x_{ki}, x_{li}]_{q^{-1}}, \qquad k > l \tag{2.1.4}$$

$$[x_{ki}, x_{lj}], \qquad k > l, i < j \tag{2.1.5}$$

$$[x_{ki}, x_{lj}] - (q^{-1} - q)x_{li}x_{kj}. \qquad k > l, i > j \tag{2.1.6}$$

The comultiplication on $K[M_q^{\mathrm{A}}]$ is given by

$$\Delta : K[M_q^{\mathrm{A}}] \to K[M_q^{\mathrm{A}}] \otimes K[M_q^{\mathrm{A}}], \quad x_{ij} \mapsto \sum_{k \in I(n)} x_{ik} \otimes x_{kj}. \tag{2.1.7}$$

Let $V = V(n)$ be the n-dimensional vector space over K with basis $\{v_i \mid i \in I(n)\}$. As a comodule V has a structure map

$$\tau_{\mathrm{A}} : V \to V \otimes K[M_q^{\mathrm{A}}], \quad v_i \mapsto \sum_j v_j \otimes x_{ji}. \tag{2.1.8}$$

For $\mu = (\mu_1, \dots, \mu_d) \in I(n)^d$, set

$$v_\mu = v_{\mu_1} \otimes \dots \otimes v_{\mu_d} \in V^{\otimes d}. \tag{2.1.9}$$

It is easy to see that the set $\{v_\mu \mid \mu \in I(n)^d\}$ forms a K-basis of the tensor space $V^{\otimes d}$. The structure map τ_{A} induces a structure map

$$\tau_{\mathrm{A}}^{\otimes d} : V^{\otimes d} \to V^{\otimes d} \otimes K[M_q^{\mathrm{A}}], \quad v_\mu \mapsto \sum_{\nu \in I(n)^d} v_\nu \otimes x_{\nu_1\mu_1} \dots x_{\nu_d\mu_d}. \tag{2.1.10}$$

In other words, the tensor space $V^{\otimes d}$ admits a $K[M_q^{\mathrm{A}}]^*$-action defined by

$$K[M_q^{\mathrm{A}}]^* \times V^{\otimes d} \to V^{\otimes d}, \quad (f, v_\mu) \mapsto \sum_{\nu \in I(n)^d} f(x_{\nu_1\mu_1} \dots x_{\nu_d\mu_d}) v_\nu. \tag{2.1.11}$$

2.2. **Hecke algebras of type B.** Let $\mathcal{H}^{\mathrm{B}} = \mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ be the two-parameter Hecke algebra of type B over K generated by $T_0, T_1, \dots, T_{d-1}$ subject to the following relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad 1 \leqslant i \leqslant d-2, \tag{2.2.1}$$

$$(T_0T_1)^2 = (T_1T_0)^2, \quad T_iT_j = T_jT_i, \qquad |i-j| > 1, \tag{2.2.2}$$

$$T_0^2 = (Q^{-1} - Q)T_0 + 1, \quad T_i^2 = (q^{-1} - q)T_i + 1, \qquad 1 \leqslant i \leqslant d-1. \tag{2.2.3}$$

That is, the corresponding Coxeter diagram is given as:

$$\underset{0}{\circ} = \underset{1}{\circ} - \cdots - \underset{d-1}{\circ}$$

Let $W^{\mathrm{B}}(d)$ be the Weyl group of type B generated by $S = \{s_0, \ldots, s_{d-1}\}$. It is known that $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ has a K-basis $\{T_w \mid w \in W^{\mathrm{B}}(d)\}$, where $T_w = T_{i_1} \ldots T_{i_N}$ for any reduced expression $w = s_{i_1} \ldots s_{i_N}$. The subalgebra of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ generated by $T_1, T_2, \ldots, T_{d-1}$ is isomorphic to the Hecke algebra $\mathcal{H}^{\mathrm{A}}_q(\Sigma_d)$ of the symmetric group Σ_d. Let $\mathcal{H}^{\mathrm{B}}_q(d)$ be the specialization of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ at $Q = q$.

2.3. **Type B Schur duality.** It is well-known that $V^{\otimes d}$ admits an $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$-action (and hence an $\mathcal{H}^{\mathrm{A}}_q(\Sigma_d)$-action) defined as follows. For $\mu = (\mu_i)_i \in I(n)^d$, $0 \leqslant t \leqslant d-1$, let

$$\mu \cdot s_t = \begin{cases} (\mu_1, \ldots, \mu_{t-1}, \mu_{t+1}, \mu_t, \mu_{t+2}, \ldots, \mu_d) & \text{if } t \neq 0; \\ (-\mu_1, \mu_2, \ldots, \mu_d) & \text{if } t = 0. \end{cases} \tag{2.3.1}$$

For $1 \leqslant t \leqslant d-1$, the right $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$-action on $V^{\otimes d}$ is defined:

$$v_\mu T_t = \begin{cases} v_{\mu \cdot s_t} & \text{if } \mu_t < \mu_{t+1}; \\ q^{-1} v_{\mu \cdot s_t} & \text{if } \mu_t = \mu_{t+1}; \\ v_{\mu \cdot s_t} + (q^{-1} - q) v_\mu & \text{if } \mu_t > \mu_{t+1}, \end{cases} \quad v_\mu T_0 = \begin{cases} v_{\mu \cdot s_0} & \text{if } 0 < \mu_1; \\ Q^{-1} v_{\mu \cdot s_0} & \text{if } 0 = \mu_1; \\ v_{\mu \cdot s_0} + (Q^{-1} - Q) v_\mu & \text{if } 0 > \mu_1. \end{cases} \tag{2.3.2}$$

The *q-Schur algebras* of type A (and B, resp.) are denoted by

$$S^{\mathrm{A}} = S^{\mathrm{A}}_q(n,d) = \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(d)}(V^{\otimes d}), \quad S^{\mathrm{B}} = S^{\mathrm{B}}_{Q,q}(n,d) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(V^{\otimes d}). \tag{2.3.3}$$

We denote by $S^{\mathrm{B}}_q(n,d)$ the specialization of $S^{\mathrm{B}}_{Q,q}(n,d)$ at $Q = q$. It is known that $S^{\mathrm{B}}_q(n,d)$ admits a geometric realization (cf. [BKLW18]) as well as a Schur duality, which is compatible with the type A duality as follows:

$$\begin{array}{ccccccccc} K[M^{\mathrm{A}}_q(n)]^* & \twoheadrightarrow & K[M^{\mathrm{A}}_q(n)]^*_d & \simeq & S^{\mathrm{A}}_q(n,d) & \curvearrowright & & \curvearrowleft & \mathcal{H}^{\mathrm{A}}(d) \\ & & & & \cup & & V^{\otimes d} & & \cap \\ & & & & S^{\mathrm{B}}_q(n,d) & \curvearrowright & & \curvearrowleft & \mathcal{H}^{\mathrm{B}}_q(d) \end{array}$$

2.4. **A coordinate coalgebra approach.** In this section, we aim to construct a coideal $J^{\mathrm{B}}_{Q,q}(n,d)$ of the coordinate bialgebra $K[M^{\mathrm{A}}_q(n)]_d$ such that $S^{\mathrm{B}}_{Q,q}(n,d)$ can be realized as the dual of the coordinate coalgebra

$$K[M^{\mathrm{B}}_{Q,q}(n)]_d = K[M^{\mathrm{A}}_q(n)]_d / J^{\mathrm{B}}_{Q,q}(n,d). \tag{2.4.1}$$

Remark 2.4.1. When it comes to comparing $S^{\mathrm{B}}_{Q,q}(n,d)$ with variants of q-Schur algebras type B (see Section 9), we call $S^{\mathrm{B}}_{Q,q}(n,d)$ a *coideal* q-Schur algebra due to this nature.

For any K-subspace J of $K[M^{\mathrm{A}}_q(n)]_d$, the $K[M^{\mathrm{A}}_q]_d$-comodule $V^{\otimes d}$ admits a $K[M^{\mathrm{A}}_q]_d/J$-comodule structure with structure map

$$\tau^{\otimes d}_J : V^{\otimes d} \to V^{\otimes d} \otimes K[M^{\mathrm{A}}_q]_d/J, \quad v_\mu \mapsto \sum_{\nu = (\nu_1, \ldots, \nu_d) \in I(n)^d} v_\nu \otimes (x_{\nu_1 \mu_1} \ldots x_{\nu_d \mu_d} + J). \tag{2.4.2}$$

We define a K-space $J^{\mathrm{B}}_{Q,q}(n,d)$ to be the intersection of all K-subspaces J satisfying that

$$(\tau^{\otimes d}_J(v_\mu))T_0 = \tau^{\otimes d}_J(v_\mu T_0), \quad \text{for all} \quad \mu \in I(n)^d. \tag{2.4.3}$$

With $J^{\mathrm{B}}_{Q,q}(n,d)$, the linear space $K[M^{\mathrm{B}}_{Q,q}(n)]_d$ is well-defined as in (2.4.1). We see from Proposition 2.4.2 that $K[M^{\mathrm{B}}_{Q,q}(n)]_d$ admits a coalgebra structure.

Proposition 2.4.2. *The K-space $K[M^{\mathrm{B}}_{Q,q}(n)]^*_d$ admits a K-algebra structure, and is isomorphic to the type B q-Schur algebra $S^{\mathrm{B}}_{Q,q}(n,d)$.*

Proof. Let Ψ be the K-algebra isomorphism $S_q^{\mathrm{A}}(n,d) \to K[M_q^{\mathrm{A}}]^*$, and hence

$$\Psi(S_{Q,q}^{\mathrm{B}}(n,d)) = \{\phi \in K[M_q^{\mathrm{A}}(n,d)]^* \mid (\phi v_\mu)T_0 = \phi(v_\mu T_0) \text{ for all } \mu \in I(n)^d\} \tag{2.4.4}$$

is a K-subalgebra of $K[M_q^{\mathrm{A}}(n,d)]^*$. By the definition of $J_{Q,q}^{\mathrm{B}}(n,d)$, as linear spaces

$$K[M_{Q,q}^{\mathrm{B}}(n)]_d^* = \{\phi \in K[M_q^{\mathrm{A}}(n,d)]^* \mid \phi(r) = 0 \text{ for all } r \in J_{Q,q}^{\mathrm{B}}(n,d)\} = \Psi(S_{Q,q}^{\mathrm{B}}(n,d)). \tag{2.4.5}$$

Hence, $K[M_{Q,q}^{\mathrm{B}}(n)]_d^*$ is isomorphic to $S_{Q,q}^{\mathrm{B}}(n,d)$ as K-subalgebras of $K[M_{Q,q}^{\mathrm{B}}(n)]_d^*$. As a consequence, the space $J_{Q,q}^{\mathrm{B}}(n,d)$ is a coideal of $K[M_{Q,q}^{\mathrm{B}}(n)]_d^*$. ☐

Let $J_{Q,q}^{\mathrm{B}}(n)$ be the union of the coideals $J_{Q,q}^{\mathrm{B}}(n,d)$ for all $d \in \mathbb{N}$, and let

$$K[M_{Q,q}^{\mathrm{B}}(n)] = K[M_q^{\mathrm{A}}(n)]/J_{Q,q}^{\mathrm{B}}(n). \tag{2.4.6}$$

Corollary 2.4.3. *The space $K[M_{Q,q}^{\mathrm{B}}(n)]$ of $K[M_q^{\mathrm{A}}(n)]$ is a quotient coalgebra.*

Proof. It follows from that $J_{Q,q}^{\mathrm{B}}(n)$ is a coideal of $K[M_q^{\mathrm{A}}(n)]$ since its degree d component $J_{Q,q}^{\mathrm{B}}(n,d)$ is a coideal of $K[M_q^{\mathrm{A}}(n)]_d$. ☐

Below we give a concrete realization of $J_{Q,q}^{\mathrm{B}}(n)$ as a right ideal. It is very important to observe that in general $J_{Q,q}^{\mathrm{B}}(n)$ is a right ideal and not a two-sided ideal, so $K[M_{Q,q}^{\mathrm{B}}(n)]$ is a coalgebra but *not* an algebra.

Proposition 2.4.4. *$J_{Q,q}^{\mathrm{B}}(n)$ is the right ideal of $K[M_q^{\mathrm{A}}(n)]$ generated by the following elements, for $i,j \in I(n)$.*

$$x_{i,j} - x_{-i,-j}, \qquad i < 0 < j \tag{2.4.7}$$
$$x_{i,j} - x_{-i,-j} - (Q^{-1} - Q)x_{-i,j}, \qquad i,j < 0 \tag{2.4.8}$$
$$x_{0,j} - Q^{-1}x_{0,-j}, \qquad j < 0 \tag{2.4.9}$$
$$x_{i,0} - Q^{-1}x_{-i,0}. \qquad i < 0 \tag{2.4.10}$$

We remark that $I(2r)$ does not contain 0 and hence $J_{Q,q}^{\mathrm{B}}(2r,d)$ is generated only by the elements of the form (2.4.7) – (2.4.8).

Proof. For a fixed $d \in \mathbb{N}$, let J be an arbitrary K-subspace of $K[M_q^{\mathrm{A}}(n)]_d$. For simplicity we write $\overline{x_{\mu\nu}} = x_{\mu\nu} + J$. For $i,j \in I(n)$ we write

$$\delta_{i<j} = \begin{cases} 1 & \text{if } i < j; \\ 0 & \text{otherwise,} \end{cases} \qquad \delta_{i>j} = \begin{cases} 1 & \text{if } i > j; \\ 0 & \text{otherwise.} \end{cases}$$

We first consider the case $d = 1$. For $i \in I(n)$,

$$\begin{aligned}
(\tau_J^{\otimes 1}(v_i))T_0 &= \sum_{j \in I(n)} v_j T_0 \otimes \overline{x_{ji}} \\
&= \sum_{j \in I(n)} \delta_{0j} Q^{-1} v_{-j} \otimes \overline{x_{ji}} + \delta_{0<j} v_{-j} \otimes \overline{x_{ji}} + \delta_{0>j}(v_{-j} + (Q^{-1} - Q)v_j) \otimes \overline{x_{ji}} \\
&= \sum_{j \in I(n)} (\delta_{0j} Q^{-1} v_j + \delta_{0<-j} v_j + \delta_{0>-j} v_j) \otimes \overline{x_{-j,i}} + \delta_{0>j}(Q^{-1} - Q)v_j) \otimes \overline{x_{ji}} \\
&= \sum_{j \in I(n)} v_j \otimes \left(\delta_{0j} Q^{-1} \overline{x_{-j,i}} + \delta_{0>j}(\overline{x_{-j,i}} + (Q^{-1} - Q)\overline{x_{ji}}) + \delta_{0<j} \overline{x_{-j,i}}\right).
\end{aligned} \tag{2.4.11}$$

On the other hand,

$$\begin{aligned}\tau_J^{\otimes 1}(v_i T_0) &= \tau_J^{\otimes 1}(\delta_{0i} Q^{-1} v_{-i} + \delta_{0<i} v_{-i} + \delta_{0>i}(v_{-i} + (Q^{-1} - Q) v_i)) \\ &= \sum_{j \in I(n)} v_j \otimes \left(\delta_{0i} Q^{-1} \overline{x_{j,-i}} + \delta_{0<i} \overline{x_{j,-i}} + \delta_{0>i} \overline{x_{j,-i}} + (Q^{-1} - Q)\overline{x_{ji}}\right).\end{aligned} \tag{2.4.12}$$

We then see that (2.4.3) holds if and only if J contains all the elements (2.4.7)–(2.4.10). Now, $J^{\mathrm{B}}_{Q,q}(n,1)$ is the linear space spanned by elements (2.4.7)–(2.4.10) since it is the intersection of all the J's satisfying (2.4.3).

For general d, since T_0 only acts on the first factor of $V^{\otimes d}$, the linear subspace $J^{\mathrm{B}}_{Q,q}(n,d)$ of $K[M^{\mathrm{A}}_q(n)]_d$ is $J^{\mathrm{B}}_{Q,q}(n,1) \otimes K[M^{\mathrm{A}}_q(n)]_{d-1}$. □

Let $\tau_B = \tau^{\otimes d}_{J^{\mathrm{B}}_{Q,q}(n,d)}$. We say that a right $K[M^{\mathrm{B}}_{Q,q}(n)]$-comodule V is *homogeneous* of degree d if all entries of its defining matrix lie in $K[M^{\mathrm{B}}_{Q,q}(n)]_d$, i.e., for a fixed basis $\{v_i\}$ of V, the $\tau_{\mathrm{B}}(v_i) = \sum_j v_j \otimes a_{ij}$ for some $a_{ij} \in K[M^{\mathrm{B}}_{Q,q}(n)]_d$.

Corollary 2.4.5. *For $d \geqslant 0$, the category of homogeneous right $K[M^{\mathrm{B}}_{Q,q}(n)]$-comodules of degree d is equivalent to the category of left $S^{\mathrm{B}}_{Q,q}(n,d)$-modules.*

2.5. **A combinatorial realization of $S^{\mathrm{B}}_{Q,q}(n,d)$.** It is well-known that the algebra $S^{\mathrm{B}}_q(n,d)$ with equal parameters admits a geometric realization via isotropic partial flags (cf. [BKLW18]). This flag realization of $S^{\mathrm{B}}_q(n,d)$ admits a combinatorial/Hecke algebraic counterpart that generalizes to a two-parameter upgrade (cf. [LL18]), i.e.,

$$S^{\mathrm{B}}_{Q,q}(n,d) = \bigoplus_{\lambda,\mu \in \Lambda^{\mathrm{B}}(n,d)} \mathrm{Hom}_{\mathcal{H}^{\mathrm{B}}_{Q,q}}(x_\mu \mathcal{H}^{\mathrm{B}}_{Q,q}, x_\lambda \mathcal{H}^{\mathrm{B}}_{Q,q}), \tag{2.5.1}$$

where

$$\Lambda^{\mathrm{B}}(n,d) = \begin{cases} \left\{\lambda = (\lambda_i)_{i \in I(n)} \in \mathbb{N}^n \;\middle|\; \lambda_0 \in 1 + 2\mathbb{Z}, \lambda_{-i} = -\lambda_i, \sum_i \lambda_i = 2d+1\right\} & \text{if } n = 2r+1; \\ \left\{\lambda = (\lambda_i)_{i \in I(n)} \in \mathbb{N}^n \;\middle|\; \lambda_{-i} = -\lambda_i, \sum_i \lambda_i = 2d\right\} & \text{if } n = 2r. \end{cases} \tag{2.5.2}$$

Note that in [LL18], the set $\Lambda^{\mathrm{B}}(2r,d)$ is identified as a subset of $\Lambda^{\mathrm{B}}(2r+1,d)$ through the embedding

$$(\lambda_i)_{i \in I(n)} \mapsto (\lambda_{-r}, \ldots, \lambda_{-1}, 1, \lambda_1, \ldots \lambda_r).$$

For any $\lambda \in \Lambda^{\mathrm{B}}(n,d)$, let W_λ be the parabolic subgroup of W^{B} generated by the set

$$\begin{cases} S - \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, \ldots, s_{\lambda_1+\ldots+\lambda_{r-1}}\} & \text{if } n = 2r; \\ S - \{s_{\lfloor \frac{\lambda_0}{2} \rfloor}, s_{\lfloor \frac{\lambda_0}{2} \rfloor + \lambda_1}, \ldots, s_{\lfloor \frac{\lambda_0}{2} \rfloor + \lambda_1 + \ldots + \lambda_{r-1}}\} & \text{if } n = 2r+1. \end{cases} \tag{2.5.3}$$

For any finite subset $X \subset W$, $\lambda, \mu \in \Lambda^{\mathrm{B}}(n,d)$ and a Weyl group element g, set

$$T_X = \sum_{w \in X} T_w, \quad T^g_{\lambda\mu} = T_{(W_\lambda)g(W_\mu)}, \quad x_\lambda = T^1_{\lambda\lambda} = T_{W_\lambda}. \tag{2.5.4}$$

The right $\mathcal{H}^{\mathrm{B}}_{Q,q}$-linear map below is well-defined:

$$\phi^g_{\lambda\mu} : x_\mu \mathcal{H}^{\mathrm{B}}_{Q,q} \to x_\lambda \mathcal{H}^{\mathrm{B}}_{Q,q}, \quad x_\mu \mapsto T^g_{\lambda\mu}. \tag{2.5.5}$$

The maps $\phi^g_{\lambda\mu}$ with $\lambda, \mu \in \Lambda^{\mathrm{B}}(n,d)$, g a minimal length double coset representative for $W_\lambda \backslash W^{\mathrm{B}} / W_\mu$ forms a linear basis for the algebra $S^{\mathrm{B}}_{Q,q}(n,d)$. The multiplication rule for $S^{\mathrm{B}}_{Q,q}(n,d)$ is given in [LL18], and it is rather involved in general. Here we only need the following facts:

Lemma 2.5.1. *Let $\lambda, \lambda', \mu, \mu' \in \Lambda^{\mathrm{B}}(n,d)$, and let g, g' be minimal length double coset representatives for $W_\lambda \backslash W^{\mathrm{B}} / W_\mu$. Then*

(a) $\phi^g_{\lambda\mu} \phi^{g'}_{\lambda'\mu'} = 0$ *unless* $\mu = \lambda'$;
(b) $\phi^1_{\lambda\mu} \phi^g_{\mu\mu'} = \phi^g_{\lambda\mu'} = \phi^g_{\lambda\mu} \phi^1_{\mu\mu'}$.

2.6. **Dimension of q-Schur algebras.** It is well-known that $S_q^{\mathrm{A}}(n,d)$ have several K-bases indexed by the set $\left\{(a_{ij})_{ij} \in \mathbb{N}^{I(n)^2} \mid \sum_{(i,j)\in I(n)^2} a_{i,j} = d\right\}$, and hence the dimension is given by

$$\dim_K S_q^{\mathrm{A}}(n,d) = \binom{n^2+d-1}{d}. \tag{2.6.1}$$

In [LL18, Lemma 2.2.1] a dimension formula is obtained via several bases of $S_{Q,q}^{\mathrm{B}}(n,d)$ with the following index set:

$$\left\{ (a_{ij})_{ij} \in \mathbb{N}^{I_-} \;\middle|\; \sum_{(i,j)\in I_-} a_{i,} = d \right\}, \quad I_- = \begin{cases} [-r,-1] \times I(n) & \text{if } n = 2r; \\ ([-r,-1] \times I(n)) \cup (\{0\} \times [-r,-1]) & \text{if } n = 2r+1. \end{cases} \tag{2.6.2}$$

That is, $I_- \subset I(n)^2$ correspond to the shaded region below:

$$\left[\begin{array}{ccc|ccc} a_{-r,-r} & & & & & a_{-r,r} \\ & \ddots & & & & \\ & & a_{-1,-1} & a_{-1,1} & & \\ \hline & & a_{1,-1} & a_{11} & & \\ & & & & \ddots & \\ a_{r,-r} & & & & & a_{rr} \end{array}\right] \qquad \left[\begin{array}{cc|c|cc} a_{-r,-r} & & & & a_{-r,r} \\ & \ddots & & & \\ \hline & & a_{00} & & \\ \hline & & & \ddots & \\ a_{r,-r} & & & & a_{rr} \end{array}\right]$$

if $n = 2r$ if $n = 2r+1$

Consequently,

$$\dim_K S_{Q,q}^{\mathrm{B}}(n,d) = \binom{|I_-|+d-1}{d} = \begin{cases} \binom{2r^2+d-1}{d} & \text{if } n = 2r; \\ \binom{2r^2+2r+d}{d} & \text{if } n = 2r+1. \end{cases} \tag{2.6.3}$$

In the following we provide a concrete description for the 2-dimensional algebra $S_{Q,q}^{\mathrm{B}}(2,1)$.

Proposition 2.6.1. *The algebra $S_{Q,q}^{\mathrm{B}}(2,1)$ is isomorphic to the type A Hecke algebra $\mathcal{H}_{Q^{-1}}(\Sigma_2)$.*

Proof. The index set here is $I(2) = \{-1,1\}$. The coalgebra $K[M_{Q,q}^{\mathrm{B}}(2)]_1$ has a K-basis $\{a = x_{-1,-1}, b = x_{-1,1} = x_{1,-1}\}$. Note that $x_{11} = a + (Q-Q^{-1})b$. The comultiplication is given by

$$\begin{aligned} \Delta(a) &= \sum_{k=\pm 1} x_{-1,k} \otimes x_{k,-1} = a \otimes a + b \otimes b, \\ \Delta(b) &= b \otimes a + (a + (Q-Q^{-1})b) \otimes b = b \otimes a + a \otimes b + (Q-Q^{-1}) b \otimes b. \end{aligned} \tag{2.6.4}$$

Hence, the algebra structure of $S_{Q,q}^{\mathrm{B}}(2,1) = K[M_{Q,q}^{\mathrm{B}}(n)]_1^*$ has a basis $\{a^*, b^*\}$ such that

$$\begin{aligned} &a^*a^*(a) = (a\otimes a)^*(\Delta(a)) = 1, \quad a^*a^*(b) = (a \otimes a)^*(\Delta(b)) = 0, \\ &a^*b^*(a) = 0 = b^*a^*(a), \quad a^*b^*(b) = 1 = b^*a^*(b), \\ &b^*b^*(a) = 1, \quad b^*b^*(b) = (Q-Q^{-1}), \end{aligned} \tag{2.6.5}$$

Therefore, the multiplication structure of $S_{Q,q}^{\mathrm{B}}(2,1)$ is given by

$$a^*a^* = a^*, \quad a^*b^* = b^* = b^*a^*, \quad b^*b^* = (Q-Q^{-1})b^* + a^*. \tag{2.6.6}$$

$\square$

Remark 2.6.2. We expect that the algebra $S_{Q,q}^{\mathrm{B}}(2,d)$ is isomorphic to $K[t]/\langle P_d(t)\rangle$ for some polynomial $P_d \in K[t]$, for $d \geqslant 1$.

3. The Isomorphism Theorem

The entire section is dedicated to the proof of an isomorphism theorem (Theorem 3.1.1) between the Schur algebras of type B and type A that is inspired by a Morita equivalence theorem due to Dipper and James [DJ92].

3.1. **The statement.** We define a polynomial $f_d^{\mathrm{B}} \in K[Q,q]$ by

$$f_d^{\mathrm{B}}(Q,q) = \prod_{i=1-d}^{d-1} (Q^{-2} + q^{2i}). \tag{3.1.1}$$

We remark that at the specialization $Q = q$, the polynomial $f_d^{\mathrm{B}}(Q,q)$ is invertible if (i) q is generic, (ii) q^2 is an odd root of unity, or (iii) q^2 is a primitive (even) ℓth root of unity for $\ell > d$.

Theorem 3.1.1. *If $f_d^{\mathrm{B}}(Q,q)$ is invertible in the field K, then we have an isomorphism of K-algebras:*

$$\Phi : S_{Q,q}^{\mathrm{B}}(n,d) \to \bigoplus_{i=0}^{d} S_q^{\mathrm{A}}(\lceil n/2 \rceil, i) \otimes S_q^{\mathrm{A}}(\lfloor n/2 \rfloor, d-i). \tag{3.1.2}$$

Example 3.1.2. For $n = 2, d = 1$, Theorem 3.1.1 gives the following isomorphism

$$S_{Q,q}^{\mathrm{B}}(2,1) \cong (S_q^{\mathrm{A}}(1,0) \otimes S_q^{\mathrm{A}}(1,1)) \oplus (S_q^{\mathrm{A}}(1,1) \otimes S_q^{\mathrm{A}}(1,0)) \cong K1_x \oplus K1_y,$$

where $1_x, 1_y$ are identities. We recall basis $\{a^*, b^*\}$ of $S_{Q,q}^{\mathrm{B}}(2,1)$ from Proposition 2.6.1. The following assignments yield the desired isomorphism:

$$a^* \mapsto 1_x + 1_y, \quad b^* \mapsto -Q^{-1}1_x + Q1_y. \tag{3.1.3}$$

We note that it remains an isomorphism if we replace $-Q^{-1}1_x + Q1_y$ in (3.1.3) by $Q1_x - Q^{-1}1_y$.

3.2. **Morita equivalence of Hecke algebras.** For now we assume that $f_d^{\mathrm{B}}(Q,q)$ is invertible. Following [DJ92], we define elements $u_i^{\pm} \in \mathcal{H}_{Q,q}^{\mathrm{B}}(d)$, for $0 \leqslant i \leqslant d$, by

$$u_i^+ = \prod_{\ell=0}^{i-1} (T_\ell \dots T_1 T_0 T_1 \dots T_\ell + Q), \quad u_i^- = \prod_{\ell=0}^{i-1} (T_\ell \dots T_1 T_0 T_1 \dots T_\ell - Q^{-1}). \tag{3.2.1}$$

It is understood that $u_0^+ = 1 = u_0^-$. For $a, b \in \mathbb{N}$ such that $a + b = d$, we define an element $v_{a,b}$ by

$$v_{a,b} = u_b^- T_{w_{a,b}} u_a^+ \in \mathcal{H}_{Q,q}^{\mathrm{B}}(d), \tag{3.2.2}$$

where $w_{a,b} \in \Sigma_{a+b}$, in two-line notation, is given by

$$w_{a,b} = \begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & a+b \\ b+1 & \cdots & b+a & 1 & \cdots & b \end{pmatrix}.$$

Finally, Dipper and James constructed an idempotent

$$e_{a,b} = \tilde{z}_{b,a}^{-1} T_{w_{b,a}} v_{a,b}, \tag{3.2.3}$$

for some invertible element $\tilde{z}_{b,a}^{-1} \in \mathcal{H}_q(\Sigma_a \times \Sigma_b)$ (see [DJ92, Definition 3.24]). Below we recall some crucial lemmas used in [DJ92].

Lemma 3.2.1. *Let $a, b \in \mathbb{N}$ be such that $a + b = d$. Then:*

(a) The elements $u_d^{\pm}$ lie in the center of $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$,

(b) $e_{a,b}\mathcal{H}_{Q,q}^{\mathrm{B}}(d)e_{a,b} = e_{a,b}\mathcal{H}_q(\Sigma_a \times \Sigma_b)$ and $e_{a,b}$ commutes with $\mathcal{H}_q(\Sigma_a \times \Sigma_b)$,

(c) $e_{a,b}\mathcal{H}_{Q,q}^{\mathrm{B}}(d) = v_{a,b}\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$,

(d) There is a Morita equivalence

$$\mathcal{H}_{Q,q}^{\mathrm{B}}(d) \underset{\mathrm{Mor}}{\simeq} \bigoplus_{i=0}^{d} e_{i,d-i}\mathcal{H}_{Q,q}^{\mathrm{B}}(d)e_{i,d-i}.$$

3.3. **The tensor space revisited.** Consider the following decompositions of V into K-subspaces:

$$V = V_{\geqslant 0} \oplus V_{<0} = V_{>0} \oplus V_{\leqslant 0},$$

where

$$V_{>0} = \bigoplus_{1\leqslant i\leqslant r} Kv_i, \quad V_{\geqslant 0} = \begin{cases} \bigoplus_{0\leqslant i\leqslant r} Kv_i, & \text{if } n = 2r+1 \\ V_{>0} & \text{if } n = 2r, \end{cases} \tag{3.3.1}$$

$$V_{<0} = \bigoplus_{-r\leqslant i\leqslant -1} Kv_i, \quad V_{\leqslant 0} = \begin{cases} \bigoplus_{-r\leqslant i\leqslant 0} Kv_i, & \text{if } n = 2r+1 \\ V_{<0} & \text{if } n = 2r. \end{cases} \tag{3.3.2}$$

Hence, one has the following canonical isomorphisms:

$$S_q^{\mathrm{A}}(\lceil n/2\rceil, d) \simeq \mathrm{End}_{\mathcal{H}_q^{\mathrm{A}}(\Sigma_d)}(V_{\geqslant 0}^{\otimes d}), \quad S_q^{\mathrm{A}}(\lfloor n/2\rfloor, d) \simeq \mathrm{End}_{\mathcal{H}_q^{\mathrm{A}}(\Sigma_d)}(V_{<0}^{\otimes d}). \tag{3.3.3}$$

In the following, we introduce two new bases $\{w_I^+\}$ and $\{w_I^-\}$ for the tensor space to help us understand the $u_d^{\pm}$-action. First define some intermediate elements, for $0 \leqslant i \leqslant r, j \in \mathbb{N}$:

$$w_{i(j)}^+ = \begin{cases} q^{-j}v_{-i} + Qv_i, & i \neq 0, \\ (q^{-2j}Q^{-1} + Q)v_i, & i = 0, \end{cases} \quad \text{and} \quad w_{i(j)}^- = \begin{cases} q^{-j}v_{-i} - Q^{-1}v_i, & i \neq 0, \\ 0, & i = 0. \end{cases} \tag{3.3.4}$$

For a nondecreasing tuple $I = (i_1, \ldots, i_d) \in ([0, r] \cap \mathbb{Z})^d$, we further define elements w_I^+ and w_I^- by

$$w_{(i)}^+ = w_{i(0)}^+, \quad w_{(i)}^- = w_{i(0)}^-, \tag{3.3.5}$$

and then inductively (on d) as below:

$$w_I^+ = w_{(i_1,\ldots,i_{d-1})}^+ \otimes w_{i_d(j)}^+, \quad w_I^- = w_{(i_1,\ldots,i_{d-1})}^- \otimes w_{i_d(j)}^-, \quad \text{where} \quad j = \max\{k : i_{d-k} = i_d\} \tag{3.3.6}$$

For arbitrary $J \in ([0, r] \cap \mathbb{Z})^d$, there is a shortest element $g \in \Sigma_d$ such that $g^{-1}J$ is nondecreasing and set

$$w_J^+ = w_{g^{-1}J}^+ T_g, \quad w_J^- = w_{g^{-1}J}^- T_g \tag{3.3.7}$$

Lemma 3.3.1. *For $I \in ([0, r] \cap \mathbb{Z})^d$,*

$$v_I u_d^+ = w_I^+, \quad v_I u_d^- = w_I^-. \tag{3.3.8}$$

Proof. For non-decreasing I, the result follows from a direct computation. For general I, there exists a shortest element $g \in \Sigma_d$ such that Ig^{-1} is non-decreasing. Then, by Lemma 3.2.1(a),

$$v_I u_d^{\pm} = v_{Ig^{-1}} T_g u_d^{\pm} = v_{Ig^{-1}} u_d^{\pm} T_g = w_{Ig^{-1}}^{\pm} T_g = w_I^{\pm}.$$ □

Example 3.3.2. Let $d = 7$ and let $I = (0, 1, 1, 2, 3, 3, 3)$. We have

$$w_I^+ = w_{0(0)}^+ \otimes w_{1(0)}^+ \otimes w_{1(1)}^+ \otimes w_{2(0)}^+ \otimes w_{3(0)}^+ \otimes w_{3(1)}^+ \otimes w_{3(2)}^+.$$

For $J := (0, 2, 1, 1, 3, 3, 3) = Is_3s_2$,

$$w_J^+ = w_I^+ T_3 T_2.$$

Example 3.3.3. In the following we verify Lemma 3.3.1 for small d's. Let $d = 2, I = (1, 1)$ and hence $w_I = w_{1(0)}^+ \otimes w_{1(1)}^+$. Since $u_2^+ = (T_1T_0T_1 + Q)(T_0 + Q)$, we can check that indeed

$$v_I u_2^+ = (v_1 \otimes v_1)(T_1T_0T_1 + Q)(T_0 + Q) = (v_1 \otimes w_{1(1)}^+)(T_0 + Q) = w_I^+.$$

Now we let $p_d : V^{\otimes d} \to V_{\leqslant 0}^{\otimes d}$ be the projection map.

Lemma 3.3.4. *For $I \in ([0, r] \cap \mathbb{Z})^d$, $J \in ([1, r] \cap \mathbb{Z})^d$, we have $p_d(w_I^+) = v_{-I}$, and $p_d(w_J^-) = v_{-J}$.*

Proof. When I, J are non-decreasing, and when $d = 2$, the result follows from a direct computation. For general I (or J), there exists a shortest element $g \in \Sigma_d$ such that Ig^{-1} (or Jg^{-1}) is non-decreasing. The result follows from an induction on the length of g. □

Now we define K-vector spaces

$$W^d_{\geqslant 0} = V^{\otimes d}_{\geqslant 0} u^+_d, \quad W^d_{<0} = V^{\otimes d}_{>0} u^-_d. \tag{3.3.9}$$

By Lemma 3.2.1(a), u^+_d and u^-_d are in the center of $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$, hence $W^d_{\geqslant 0}$ and $W^d_{<0}$ are naturally $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$-module via right multiplication.

Lemma 3.3.5. *(a) The map $V^{\otimes d}_{\geqslant 0} \to W^d_{\geqslant 0}$ given by the right multiplication of u^+_d is an isomorphism of $\mathcal{H}^{\mathrm{A}}_q(\Sigma_d)$-modules, and $wT_0 = Q^{-1}w$ for all $w \in W^d_{\geqslant 0}$,*

(b) The map $V^{\otimes d}_{<0} \to W^d_{<0}$ given by the right multiplication of u^-_d is an isomorphism of $\mathcal{H}^{\mathrm{A}}_q(\Sigma_d)$-modules, and $wT_0 = -Qw$ for all $w \in W^d_{<0}$,

Proof. By Lemma 3.3.4, the maps $v_I \mapsto p_d(v_I u^\pm_d)$ give isomorphisms of vector spaces

$$V^{\otimes d}_{\geqslant 0} \cong p_d(W^d_{\geqslant 0}) \quad \text{and} \quad V^{\otimes d}_{<0} \cong p_d(W^d_{<0}).$$

So, the right multiplications by $u^\pm_d$ are isomorphisms of vector spaces. It then follows from Lemma 3.2.1 that they are isomorphism of $\mathcal{H}^{\mathrm{A}}_q(\Sigma_d)$-modules. The T_0 action follows from a direct computation. □

3.4. **The actions of $T_{w_{a,b}}$ and $v_{a,b}$.** For $a+b=d$, consider the projections

$$p_{a,b} : V^{\otimes d} \to V^{\otimes a}_{\leqslant 0} \otimes V^{\otimes b}_{<0}, \quad p'_{a,b} : V^{\otimes d} \to V^{\otimes a} \otimes V^{\otimes b}_{<0}.$$

Lemma 3.4.1. *Let $I \in ([0,r] \cap \mathbb{Z})^a$, $J \in ([-r,r] \cap \mathbb{Z})^b$. Then $p'_{a,b}((v_J \otimes v_I)T_{w_{a,b}}) = v_I \otimes p_b(v_J)$.*

Proof. First note that $(v_J \otimes v_I)T_{w_{a,b}} = (v_J \otimes v_I)w_{a,b} + \sum_{g<w_{a,b}} c_{a,b}(v_J \otimes v_I)g)$ for some $c_{a,b} \in K$, where $g < w_{a,b}$ under the Bruhat order. Hence,

$$\begin{aligned} p'_{a,b}((v_J \otimes v_I)T_{w_{a,b}}) &= p'_{a,b}((v_J \otimes v_I)w_{a,b} + \sum_{g<w_{a,b}} c_{a,b}(v_J \otimes v_I)g) \\ &= p'_{a,b}(v_I \otimes v_J) + \sum_{g<w_{a,b}} c_{a,b} p'_{a,b}((v_J \otimes v_I)g) = v_I \otimes p_b(v_J). \end{aligned}$$

□

Lemma 3.4.2. *For $a+b=d$, the following map f is an isomorphism of $\mathcal{H}^{\mathrm{A}}(\Sigma_a) \otimes \mathcal{H}^{\mathrm{A}}(\Sigma_b)$-module.*

$$f : V^{\otimes a}_{\geqslant 0} \otimes V^{\otimes b}_{>0} \to W^a_{\geqslant 0} \otimes W^b_{<0}, \quad v_I \otimes v_J \mapsto (v_J \otimes v_I)v_{a,b}.$$

Proof. For $I \in [0,r]^d$ and $J \in [1,r]^d$, we have

$$\begin{aligned} p_{a,b}((v_J \otimes v_I)v_{a,b}) &= p_{a,b}((v_J \otimes v_I)u^-_b T_{w_{a,b}} u^+_a) && \text{Eq. (3.2.2)} \\ &= p_{a,b}((w^-_J \otimes v_I)T_{w_{a,b}} u^+_a) && \text{Lemma 3.3.1} \\ &= p_{a,b}(p'_{a,b}((w^-_J \otimes v_I)T_{w_{a,b}})u^+_a) && \\ &= p_{a,b}((v_I \otimes p_b(w^-_J))u^+_a) && \text{Lemma 3.4.1} \\ &= p_{a,b}(w^+_I \otimes p_b(w^-_J)) && \text{Lemma 3.3.1} \\ &= p_a(w^+_I) \otimes p_b(w^-_J) && \\ &= v_{-I} \otimes v_{-J}. && \text{Lemma 3.3.4} \end{aligned}$$

Therefore, $p_{a,b} \circ f$ is an epimorphism of vector space $V^{\otimes b}_{\geqslant 0} \otimes V^{\otimes a}_{>0} \to V^{\otimes a}_{\leqslant 0} \otimes V^{\otimes b}_{<0}$. By dimension counting, it is an isomorphism, hence f is an isomorphism as well. □

3.5. **The proof.** Finally, we are in a position to prove the isomorphism theorem.

Proof of Theorem 3.1.1. We first show that $S^{\mathrm{B}}_{Q,q}(n,d) \supseteq \bigoplus_i S^{\mathrm{A}}_q(\lceil n/2\rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i)$ as below:

$$\begin{aligned}
S^{\mathrm{B}}_{Q,q}(n,d) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(V^{\otimes d}) &= \mathrm{End}_{\bigoplus_{0\leqslant i\leqslant d} e_{i,d-i}\mathcal{H}^{\mathrm{B}}_{Q,q}(d)e_{i,d-i}}(V^{\otimes d}e_{i,d-i}) && \text{Lemma 3.2.1(d)}\\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{e_{i,d-i}\mathcal{H}^{\mathrm{B}}_{Q,q}(d)e_{i,d-i}}(V^{\otimes d}e_{i,d-i})\\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)\otimes\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}(V^{\otimes d}v_{i,d-i}) && \text{Lemma 3.2.1(b)(c)}\\
&\supseteq \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)\otimes\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}((V^{\otimes d-i}_{>0}\otimes V^{\otimes i}_{\geqslant 0})v_{i,d-i})\\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)\otimes\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}(W^{i}_{\geqslant 0}\otimes W^{d-i}_{<0}) && \text{Lemma 3.4.2}\\
&= \bigoplus_{0\leqslant i\leqslant d} \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_i)}(V^{\otimes i}_{\geqslant 0})\otimes \mathrm{End}_{\mathcal{H}^{\mathrm{A}}_q(\Sigma_{d-i})}(V^{\otimes d-i}_{>0}) && \text{Lemma 3.3.5(a)}\\
&= \bigoplus_{0\leqslant i\leqslant d} S^{\mathrm{A}}_q(\lceil n/2\rceil, i)\otimes S^{\mathrm{A}}_q(\lfloor n/2\rfloor, d-i). && \text{Eq. (3.3.3)}
\end{aligned}$$

The theorem follows by a dimension comparison using (2.6.1) and (2.6.3). □

3.6. As an immediate consequence of the Morita equivalence theorem one obtains a classification of irreducible representations for $S^{\mathrm{B}}_{Q,q}(n,d)$.

Theorem 3.6.1. *If $f^{\mathrm{B}}_d(Q,q)$ is invertible in the field K then there is a bijection*

$$\{\text{Irreducible representations of } S^{\mathrm{B}}_{Q,q}(n,d)\} \leftrightarrow \{(\lambda,\mu)\vdash(d_1,d_2) \mid d_1+d_2=d\},$$

where number of parts of λ and μ is no more than n. In particular, the standard modules over $S^{\mathrm{B}}_{Q,q}(n,d)$ are of the form $\nabla(\lambda)\boxtimes\nabla(\mu)$, where $\nabla(\lambda)$ (resp. $\nabla(\lambda)$) are standard modules over $S^{\mathrm{A}}_q(n,d_1)$ (resp. $S^{\mathrm{A}}_q(n,d_2)$).

4. Schur Functors

4.1. **Schur functors.** For type A it is well-known that, provided $n \geqslant d$, there is an idempotent $e^{\mathrm{A}} = e^{\mathrm{A}}(n,d) \in S^{\mathrm{A}}_q(n,d)$ such that $e^{\mathrm{A}}S^{\mathrm{A}}_q(n,d)e^{\mathrm{A}} \simeq \mathcal{H}_q(\Sigma_d)$, and a Schur functor

$$F^{\mathrm{A}}_{n,d} : \mathrm{Mod}(S^{\mathrm{A}}_q(n,d)) \to \mathrm{Mod}(\mathcal{H}_q(\Sigma_d)), \quad M \mapsto e^{\mathrm{A}}M. \tag{4.1.1}$$

In the following proposition we construct the Schur functor for $S^{\mathrm{B}}_{Q,q}(n,d)$ when $\lfloor \frac{n}{2}\rfloor \geqslant d$.

Proposition 4.1.1. *If $\lfloor \frac{n}{2}\rfloor \geqslant d$ then there is an idempotent $e^{\mathrm{B}} = e^{\mathrm{B}}(n,d) \in S^{\mathrm{B}}_{Q,q}(n,d)$ such that $e^{\mathrm{B}}S^{\mathrm{B}}_{Q,q}(n,d)e^{\mathrm{B}} \simeq \mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ as K-algebras, and $e^{\mathrm{B}}S^{\mathrm{B}}_{Q,q}(n,d) \simeq V^{\otimes d}$ as $(S^{\mathrm{B}}_{Q,q}(n,d), \mathcal{H}^{\mathrm{B}}_{Q,q}(d))$-bimodules..*

Proof. Recall $\Lambda^{\mathrm{B}}(n,d)$ from (2.5.2) and $\phi^g_{\lambda\mu}$ from (2.5.5). Let $e^{\mathrm{B}} = \phi^1_{\omega\omega}$, where

$$\omega = \begin{cases} \{(0,\dots,0,\underbrace{1,\dots,1}_{2d},0\dots,0) \in \Lambda^{\mathrm{B}}(2r,d)\} & \text{if } n = 2r;\\ \{(0,\dots,0,\underbrace{1,\dots,1}_{2d+1},0\dots,0) \in \Lambda^{\mathrm{B}}(2r+1,d)\} & \text{if } n = 2r+1. \end{cases} \tag{4.1.2}$$

Note that such ω is well-defined only when $r = \lfloor \frac{n}{2}\rfloor \geqslant d$. By Lemma 2.5.1, we have

$$e^{\mathrm{B}}\phi^g_{\lambda\mu}e^{\mathrm{B}} = \begin{cases} \phi^g_{\lambda\mu} & \text{if } \lambda = \omega = \mu;\\ 0 & \text{otherwise.} \end{cases} \tag{4.1.3}$$

Since W_ω is the trivial group, $x_\omega = 1 \in \mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ and hence $\phi^g_{\omega\omega}$ is uniquely determined by $1 \mapsto T_g$. Therefore, $e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d) e^{\mathrm{B}}$ and $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ are isomorphic as algebras.

Now from Section 2.5 we see that there is a canonical identification

$$V^{\otimes d} \simeq \bigoplus_{\mu \in \Lambda^{\mathrm{B}}(n,d)} x_\mu \mathcal{H}^{\mathrm{B}}_{Q,q} \simeq \bigoplus_{\mu \in \Lambda^{\mathrm{B}}(n,d)} \mathrm{Hom}_{\mathcal{H}^{\mathrm{B}}_{Q,q}}(x_\omega \mathcal{H}^{\mathrm{B}}_{Q,q}, x_\mu \mathcal{H}^{\mathrm{B}}_{Q,q}), \tag{4.1.4}$$

and hence the maps $\phi^g_{\omega\mu}$, with $\mu \in \Lambda^{\mathrm{B}}(n,d)$, g a minimal length coset representative for W^{B}/W_μ, forms a linear basis for $V^{\otimes d}$. Again by Lemma 2.5.1, we have

$$e^{\mathrm{B}} \phi^g_{\lambda\mu} = \begin{cases} \phi^g_{\omega\mu} & \text{if } \lambda = \omega; \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.5}$$

Hence, $e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d)$ has a linear basis $\{\phi^g_{\omega\mu}\}$ where $\mu \in \Lambda^{\mathrm{B}}(n,d)$, g a minimal length double coset representative for $W_\omega \backslash W^{\mathrm{B}}/W_\mu$. Therefore $V^{\otimes d}$ and $e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d)$ are isomorphic as $(S^{\mathrm{B}}_{Q,q}(n,d), \mathcal{H}^{\mathrm{B}}_{Q,q}(d))$-bimodules. □

We define the Schur functor of type B by

$$F^{\mathrm{B}}_{n,d} : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)), \quad M \mapsto e^{\mathrm{B}} M. \tag{4.1.6}$$

Define the inverse Schur functor by

$$G^{\mathrm{B}}_d : \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)) \to \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)), \quad M \mapsto \mathrm{Hom}_{e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d) e^{\mathrm{B}}}(e^{\mathrm{B}} S^{\mathrm{B}}_{Q,q}(n,d), M). \tag{4.1.7}$$

In below we define a Schur-like functor $F^\flat_{n,d} : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d))$ using Theorem 3.1.1, under the same invertibility assumption: recall Φ from (3.1.2), let

$$\epsilon^\flat = \epsilon^\flat_{n,d} = \Phi^{-1}\Big(\bigoplus_{i=0}^{d} e^{\mathrm{A}}(\lceil \tfrac{n}{2} \rceil, i) \otimes e^{\mathrm{A}}(\lfloor \tfrac{n}{2} \rfloor, d-i)\Big). \tag{4.1.8}$$

Note that $\epsilon^\flat S^{\mathrm{B}}_{Q,q}(n,d) \epsilon^\flat \simeq \bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1})$, and hence left multiplication by $\epsilon^\flat$ defines a functor $\mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1}))$. Hence, we can define

$$F^\flat_{n,d} : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)), \quad M \mapsto \mathcal{F}_H^{-1}(\epsilon^\flat M), \tag{4.1.9}$$

where $\mathcal{F}_H$ is the Morita equivalence for the Hecke algebras given by

$$\mathcal{F}_H : \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)) \to \mathrm{Mod}\Big(\bigoplus_{i} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1})\Big). \tag{4.1.10}$$

Under the invertibility condition, one can define an equivalence of categories induced from Φ as below:

$$\mathcal{F}_S : \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) \to \mathrm{Mod}\Big(\bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil \tfrac{n}{2} \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor \tfrac{n}{2} \rfloor, d-i)\Big). \tag{4.1.11}$$

In other words, we have the following commutativity of functors:

Proposition 4.1.2. *Assume $\lfloor \frac{n}{2} \rfloor \geqslant d \geqslant i \geqslant 0$ and that f^{B}_q is invertible. The diagram below commutes:*

$$\begin{array}{ccc} \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d)) & \xrightarrow{\mathcal{F}_s} & \mathrm{Mod}\Big(\bigoplus_{i=0}^{d} S^{\mathrm{A}}_q(\lceil \tfrac{n}{2} \rceil, i) \otimes S^{\mathrm{A}}_q(\lfloor \tfrac{n}{2} \rfloor, d-i)\Big) \\ \Big\downarrow {\scriptstyle F^\flat_{n,d}} & & \Big\downarrow {\scriptstyle \bigoplus_{i=0}^{d} F^{\mathrm{A}}_{\lceil \frac{n}{2} \rceil, i} \otimes F^{\mathrm{A}}_{\lfloor \frac{n}{2} \rfloor, d-i}} \\ \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d)) & \xrightarrow{\mathcal{F}_H} & \mathrm{Mod}\Big(\bigoplus_{i=0}^{d} \mathcal{H}_q(\Sigma_{i+1}) \otimes \mathcal{H}_q(\Sigma_{d-i+1})\Big) \end{array} \tag{4.1.12}$$

Remark 4.1.3. We expect that Proposition 4.1.2 still holds if we replace the functor $F^\flat_{n,d}$ therein by $F^{\mathrm{B}}_{n,d}$.

4.2. **Existence of idempotents.** We construct additional idempotents in Schur algebras of type B that will be used later in Section 7.

Proposition 4.2.1. *There exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n',d)$ such that $eS^{\mathrm{B}}_{Q,q}(n',d)e \simeq S^{\mathrm{B}}_{Q,q}(n,d)$ if either one of the following holds:*

(a) $n' \geqslant n$ and $n' \equiv n \mod 2$;
(b) $n' = 2r' + 1 \geqslant n = 2r$.

Proof. We use the combinatorial realization in Section 2.5. For (a) we set

$$e = \sum_{\gamma} \phi^1_{\gamma\gamma}, \tag{4.2.1}$$

where γ runs over the set

$$\Lambda^{\mathrm{B}}(n',d)|_n = \begin{cases} \{\gamma = (0,\dots,0,\underbrace{*,\dots,*}_{n},0\dots,0) \in \Lambda^{\mathrm{B}}(n',d)\} & \text{if } (a) \text{ holds;} \\ \{\gamma = (0,\dots,0,\underbrace{*,\dots,*}_{r},1,\underbrace{*,\dots,*}_{r},0\dots,0) \in \Lambda^{\mathrm{B}}(n',d)\} & \text{if } (b) \text{ holds.} \end{cases} \tag{4.2.2}$$

By Lemma 2.5.1 we have

$$e\phi^g_{\lambda\mu}e = \begin{cases} \phi^g_{\lambda\mu} & \text{if } \lambda,\mu \in \Lambda^{\mathrm{B}}(n',d)|_n; \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.3}$$

It follows by construction that $eS^{\mathrm{B}}_{Q,q}(n',d)e$ and $S^{\mathrm{B}}_{Q,q}(n,d)$ are isomorphic as algebras. □

4.3. **Existence of spectral sequences.** Let A be a finite dimensional algebra over a field k and e be an idempotent in A. Doty, Erdmann and Nakano [DEN04] established a relationship between the cohomology theory in $\mathrm{Mod}(A)$ versus $\mathrm{Mod}(eAe)$. More specifically, they construct a Grothendieck spectral sequence which starts from extensions of A-modules and converges to extensions of eAe-modules.

There are two important functors involved in this construction. The first functor is an exact functor from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(eAe)$ denoted by $\mathcal{F}$ (that is a special case of the classical Schur functor) defined by $\mathcal{F}(-) = e(-)$. The other functor is a left exact functor from $\mathrm{Mod}(eAe)$ to $\mathrm{Mod}(A)$, denoted $\mathcal{G}$ defined by $\mathcal{G}(-) = \mathrm{Hom}_A(Ae,-)$. This functor is right adjoint to $\mathcal{F}$.

In [DEN04], the aforementioned construction was used in the quantum setting to relate the extensions for quantum GL_n to those for Hecke algebras. For $\lfloor \frac{n}{2} \rfloor \geqslant d$ there exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n,d)$ such that $\mathcal{H}^{\mathrm{B}}_{Q,q}(d) \cong eS^{\mathrm{B}}_{Q,q}(n,d)e$. Therefore, we obtain a relationship between cohomology of the type B Schur algebras with the Hecke algebra of type B.

Theorem 4.3.1. *Let $\lfloor \frac{n}{2} \rfloor \geqslant d$ with $M \in \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d))$ and $N \in \mathrm{Mod}(\mathcal{H}^{\mathrm{B}}_{Q,q}(d))$. There exists a first quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}^i_{S^{\mathrm{B}}_{Q,q}(n,d)}(M, R^j\mathcal{G}(N)) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(eM,N).$$

where $R^j\mathcal{G}(-) = \mathrm{Ext}^j_{\mathcal{H}^{\mathrm{B}}_{Q,q}(d)}(V^{\otimes d}, N)$.

We can also compare cohomology between $S^{\mathrm{B}}_{Q,q}(n,d)$ and $S^{\mathrm{B}}_{Q,q}(n',d)$ where $n' \geqslant n$ since there exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n',d)$ such that $S^{\mathrm{B}}_{Q,q}(n,d) \cong eS^{\mathrm{B}}_{Q,q}(n',d)e$ thanks to Proposition 4.2.1.

Theorem 4.3.2. *Let $M \in \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n',d))$ and $N \in \mathrm{Mod}(S^{\mathrm{B}}_{Q,q}(n,d))$. Assume that either*

(a) $n' \geqslant n$ and $n' \equiv n \mod 2$;
(b) $n' = 2r' + 1 \geqslant n = 2r$.

Then there exists a first quadrant spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_{S^{\mathrm{B}}_{Q,q}(n',d)}(M, R^j\mathcal{G}(N)) \Rightarrow \operatorname{Ext}^{i+j}_{S^{\mathrm{B}}_{Q,q}(n,d)}(eM, N).$$

where $R^j\mathcal{G}(-) = \operatorname{Ext}^j_{S^{\mathrm{B}}_{Q,q}(n,d)}(eS^{\mathrm{B}}_{Q,q}(n',d), -)$.

5. Cellularity

5.1. **Definition.** We start from recalling the definition of a cellular algebra following [GL96]. A K-algebra A is *cellular* if it is equipped with a cell datum $(\Lambda, M, C, *)$ consisting of a poset Λ, a map M sending each $\lambda \in \Lambda$ to a finite set $M(\lambda)$, a map C sending each pair $(\mathfrak{s}, \mathfrak{t}) \in M(\lambda)^2$ to an element $C^\lambda_{\mathfrak{s},\mathfrak{t}} \in A$, and an K-linear involutory anti-automorphism $*$ satisfying the following conditions:

(C1) The map C is injective with image being an K-basis of A (called a *cellular basis*).
(C2) For any $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, $(C^\lambda_{\mathfrak{s},\mathfrak{t}})^* = C^\lambda_{\mathfrak{t},\mathfrak{s}}$.
(C3) There exists $r_a(\mathfrak{s}', \mathfrak{s}) \in K$ for $\lambda \in \Lambda, \mathfrak{s}, \mathfrak{s}' \in M(\lambda)$ such that for all $a \in A$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$,

$$aC^\lambda_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{s}' \in M(\lambda)} r_a(\mathfrak{s}', \mathfrak{s}) C^\lambda_{\mathfrak{s}',\mathfrak{t}} \mod A_{<\lambda}.$$

Here $A_{<\lambda}$ is the K-submodule of A generated by the set $\{C^\mu_{\mathfrak{s}'',\mathfrak{t}''} \mid \mu < \lambda; \mathfrak{s}'', \mathfrak{t}'' \in M(\mu)\}$.

For a cellular algebra A, we define for each $\lambda \in \Lambda$ a *cell module* $W(\lambda)$ spanned by $C^\lambda_{\mathfrak{s}}$, $\mathfrak{s} \in M(\lambda)$, with multiplication given by

$$aC_{\mathfrak{s}} = \sum_{\mathfrak{s}' \in M(\lambda)} r_a(\mathfrak{s}', \mathfrak{s}) C'_{\mathfrak{s}}. \tag{5.1.1}$$

For each $\lambda \in \Lambda$ we let $\phi_\lambda : W(\lambda) \times W(\lambda) \to K$ be a bilinear form satisfying

$$C^\lambda_{\mathfrak{s},\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{t}} \equiv \phi_\lambda(C_{\mathfrak{s}}, C_{\mathfrak{t}}) C^\lambda_{\mathfrak{s},\mathfrak{t}} \mod A_{<\lambda}. \tag{5.1.2}$$

It is known that the type A q-Schur algebras are always cellular, and there could be distinct cellular structures. See [AST18] for a parallel approach on the cellularity of centralizer algebras for quantum groups.

Example 5.1.1 (Mathas)**.** Let $\Lambda = \Lambda^{\mathrm{A}}(d)$ be the set of all partitions of d, and let $\Lambda' = \Lambda'(d)$ be the set of all compositions of d. For each composition $\lambda \in \Lambda'$, let Σ_λ be the corresponding Young subgroup of Σ_d. We set

$$x_\lambda = \sum_{w \in \Sigma_\lambda} T_w \in \mathcal{H}_q(\Sigma_d).$$

It is known the q-Schur algebra admits the following combinatorial realization:

$$S^{\mathrm{A}}_q(n,d) = \operatorname{End}_{\mathcal{H}_q(\Sigma_d)}(\oplus_{\lambda \in \Lambda'} x_\lambda \mathcal{H}_q(\Sigma_d)) = \bigoplus_{\lambda,\mu \in \Lambda'} \operatorname{Hom}_{\mathcal{H}_q(\Sigma_d)}(x_\mu \mathcal{H}_q(\Sigma_d), x_\lambda \mathcal{H}_q(\Sigma_d)).$$

The finite set $M(\lambda)$ is given by $M(\lambda) = \bigsqcup_{\mu \in \Lambda'} \mathrm{SSTD}(\lambda, \mu)$, where

$$\mathrm{SSTD}(\lambda, \mu) = \{\text{semi-standard } \lambda\text{-tableaux of shape } \mu\}. \tag{5.1.3}$$

For $\mu \vdash d$, denote the set of shortest right coset representatives for Σ_μ in Σ_d by

$$D_\mu = \{w \in \Sigma_d \mid \ell(gw) = \ell(w) + \ell(g) \text{ for all } g \in \Sigma_\mu\}. \tag{5.1.4}$$

Let $\mathfrak{t}^\lambda$ be the canonical λ-tableau of shape λ, then for all λ-tableau $\mathfrak{t}$ there is a unique element $d(\mathfrak{t}) \in D_\lambda$ such that $\mathfrak{t}d(\mathfrak{t}) = \mathfrak{t}$. The cellular basis element, for $\lambda \in \Lambda, \mathfrak{s} \in \mathrm{sstd}(\lambda, \mu), \mathfrak{t} \in \mathrm{sstd}(\lambda, \nu)$, is the given by

$$C^\lambda_{\mathfrak{s},\mathfrak{t}}(x_\alpha h) = \delta_{\alpha,\mu} \sum_{s,t} T_{d(s)^{-1}} x_\lambda T_{d(t)} h, \tag{5.1.5}$$

where the sum is over all pairs (s, t) such that $\mu(s) = \mathfrak{s}, \nu(t) = \mathfrak{t}$.

Example 5.1.2 (Doty-Giaquinto)**.** The poset Λ is the same as in Example 5.1.1, and we have $\Lambda = \Sigma_d\Lambda^+$. It is known that the algebra $S_q^{\mathrm{A}}(n,d)$ admits a presentation with generators $E_i, F_i (1 \leqslant i \leqslant n-1)$ and $1_\lambda (\lambda \in \Lambda)$. The map $*$ is the anti-automorphism satisfying

$$E_i^* = F_i, \quad F_i^* = E_i, \quad 1_\lambda^* = 1_\lambda.$$

For each $\lambda \in \Lambda$ we set $\Lambda_\lambda^+ = \{\mu \in \Lambda^+ \mid \mu \leqslant \lambda\}$. Note that Λ_λ^+ is saturated and it defines a subalgebra $S_q(\Lambda_\lambda^+)$ of $S_q^{\mathrm{A}}(n,d)$ with a basis $\{\overline{x}_{\mathfrak{s}} \mid 1 \leqslant \mathfrak{s} \leqslant d_\lambda\}$ for some $d_\lambda \in \mathbb{N}$. Let $x_{\mathfrak{s}} \in S_q^{\mathrm{A}}(n,d)^-$ be the preimage of $\overline{x}_{\mathfrak{s}}$ under the projection $S_q^{\mathrm{A}}(n,d) \to S_q(\Lambda_\lambda^+)$ that is the identity map except for that it kills all 1_μ where $\mu \nleqslant \lambda$. The finite set $M(\lambda)$ is given by

$$M(\lambda) = \{1, 2, \dots, d_\lambda\}. \tag{5.1.6}$$

Finally, for $\lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in M(\lambda)$, we set

$$C_{\mathfrak{s},\mathfrak{t}}^\lambda = x_{\mathfrak{s}} 1_\lambda x_{\mathfrak{t}}^*. \tag{5.1.7}$$

5.2. **Cellular structures on $S_{Q,q}^{\mathrm{B}}(n,d)$.** We show that the isomorphism theorem produces a cellular structure for $S_{Q,q}^{\mathrm{B}}(n,d)$ using any cellular structure on the q-Schur algebras of type A. For any n, d we fix a cell datum $(\Lambda_{n,d}, M_{n,d}, C_{n,d}, *)$ for $S_q^{\mathrm{A}}(n,d)$. Define

$$\Lambda^{\mathrm{B}} = \Lambda^{\mathrm{B}}(n,d) = \bigsqcup_{i=0}^{d} \Lambda_{\lceil \frac{n}{2} \rceil, i} \times \Lambda_{\lfloor \frac{n}{2} \rfloor, d-i}, \tag{5.2.1}$$

as a poset with the lexicographical order. For $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^{\mathrm{B}}$, we define M^{B} by

$$M^{\mathrm{B}}(\lambda) = \bigsqcup_{i=0}^{d} M_{\lceil \frac{n}{2} \rceil, i}(\lambda^{(1)}) \times M_{\lfloor \frac{n}{2} \rfloor, d-i}(\lambda^{(2)}). \tag{5.2.2}$$

The map C^{B} is given by, for $\mathfrak{s} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)}), \mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) \in M_{\lceil \frac{n}{2} \rceil, i}(\lambda^{(1)}) \times M_{\lfloor \frac{n}{2} \rfloor, d-i}(\lambda^{(2)}) \subset M^{\mathrm{B}}(\lambda)$,

$$(C^{\mathrm{B}})_{\mathfrak{s},\mathfrak{t}}^\lambda = (C_{\lceil \frac{n}{2} \rceil, i})_{\mathfrak{s}^{(1)},\mathfrak{t}^{(1)}}^{\lambda^{(1)}} \otimes (C_{\lfloor \frac{n}{2} \rfloor, d-i})_{\mathfrak{s}^{(2)},\mathfrak{t}^{(2)}}^{\lambda^{(2)}}. \tag{5.2.3}$$

Finally, the map $*$ is given by

$$* : (C_{\lceil \frac{n}{2} \rceil, i})_{\mathfrak{s}^{(1)},\mathfrak{t}^{(1)}}^{\lambda^{(1)}} \otimes (C_{\lfloor \frac{n}{2} \rfloor, d-i})_{\mathfrak{s}^{(2)},\mathfrak{t}^{(2)}}^{\lambda^{(2)}} \mapsto (C_{\lceil \frac{n}{2} \rceil, i})_{\mathfrak{t}^{(1)},\mathfrak{s}^{(1)}}^{\lambda^{(1)}} \otimes (C_{\lfloor \frac{n}{2} \rfloor, d-i})_{\mathfrak{t}^{(2)},\mathfrak{s}^{(2)}}^{\lambda^{(2)}}. \tag{5.2.4}$$

Corollary 5.2.1. *If the invertibility condition in Theorem 3.1.1 holds, then $S_{Q,q}^{\mathrm{B}}(n,d)$ is a cellular algebra with cell datum $(\Lambda^{\mathrm{B}}, M^{\mathrm{B}}, C^{\mathrm{B}}, *)$.*

Proof. Condition (C1) follows from the isomorphism theorem; while Condition (C2) follows directly from (5.2.4). Condition (C3) follows from the type A cellular structure as follows: for $a_1 \in S_q^{\mathrm{A}}(\lceil \frac{n}{2} \rceil, i)$ and $a_2 \in S_q^{\mathrm{A}}(\lfloor \frac{n}{2} \rfloor, d-i)$,

$$a_1 (C_{\lceil \frac{n}{2} \rceil, i})_{\mathfrak{s}^{(1)},\mathfrak{t}^{(1)}}^{\lambda^{(1)}} \equiv \sum_{\mathfrak{u}^{(1)} \in M_{\lceil \frac{n}{2} \rceil, i}(\lambda^{(1)})} r_{a_1}^{(1)}(\mathfrak{u}^{(1)}, \mathfrak{s}^{(1)}) (C_{\lceil \frac{n}{2} \rceil, i})_{\mathfrak{u}^{(1)},\mathfrak{t}^{(1)}}^{\lambda^{(1)}} \mod S_q^{\mathrm{A}}(\lceil \tfrac{n}{2} \rceil, i)(< \lambda^{(1)}),$$

$$a_2 (C_{\lfloor \frac{n}{2} \rfloor, d-i})_{\mathfrak{s}^{(2)},\mathfrak{t}^{(2)}}^{\lambda^{(2)}} \equiv \sum_{\mathfrak{u}^{(2)} \in M_{r, d-i}(\lambda^{(2)})} r_{a_2}^{(2)}(\mathfrak{u}^{(2)}, \mathfrak{s}^{(2)}) (C_{\lfloor \frac{n}{2} \rfloor, d-i})_{\mathfrak{u}^{(2)},\mathfrak{t}^{(2)}}^{\lambda^{(2)}} \mod S_q^{\mathrm{A}}(\lfloor \tfrac{n}{2} \rfloor, d-i)(< \lambda^{(2)}).$$

That is, for $a = a_1 \otimes a_2 \in S_q^{\mathrm{A}}(\lceil \frac{n}{2} \rceil, i) \otimes S_q^{\mathrm{A}}(\lfloor \frac{n}{2} \rfloor, d-i) \subset S_q^{\mathrm{B}}(n,d)$ we have

$$a (C^{\mathrm{B}})_{\mathfrak{s},\mathfrak{t}}^\lambda \equiv \sum_{\substack{\mathfrak{u} = (\mathfrak{u}^{(1)}, \mathfrak{u}^{(2)}) \\ \in M_{\lceil \frac{n}{2} \rceil, i}(\lambda^{(1)}) \times M_{\lfloor \frac{n}{2} \rfloor, d-i}(\lambda^{(2)})}} r_a^{\mathrm{B}}(\mathfrak{u}, \mathfrak{s}) (C^{\mathrm{B}})_{\mathfrak{u},\mathfrak{t}}^\lambda \mod S_q^{\mathrm{B}}(n,d)(< \lambda),$$

where $r_a^{\mathrm{B}}(\mathfrak{u}, \mathfrak{s}) = r_{a_1}^{(1)}(\mathfrak{u}^{(1)}, \mathfrak{s}^{(1)}) r_{a_2}^{(2)}(\mathfrak{u}^{(2)}, \mathfrak{s}^{(2)})$ is independent of $\mathfrak{t}$. $\square$

6. Quasi-Hereditary Structure

6.1. **Definition.** Following [CPS88], a K-algebra A is called quasi-hereditary if there is a chain of two-sided ideals of A:

$$0 \subset I_1 \subset I_2 \subset \ldots \subset I_n = A$$

such that each quotient $J_j = I_j/I_{j-1}$ is a hereditary ideal of A/I_{j-1}. It is known [GL96] that if A is cellular and $\phi_\lambda \neq 0$ (cf. (5.1.2)) for all $\lambda \in \Lambda$ then A is quasi-hereditary.

An immediate corollary of our isomorphism theorem is that $S^{\mathrm{B}}_{Q,q}(n,d)$ is quasi-hereditary under the invertibility condition. We conjecture that this is a sufficient and necessary condition and provide some evidence for small n.

Corollary 6.1.1. *If the invertibility condition in Theorem 3.1.1 holds, then $S^{\mathrm{B}}_q(n,d)$ is quasi-hereditary.*

Proof. Let ϕ^{A}_ν with $\nu \in \Lambda_{r,j}$ be such a map for $S^{\mathrm{A}}_q(r,j)$. Fix $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_{\lceil \frac{n}{2} \rceil, i} \times \Lambda_{\lfloor \frac{n}{2} \rfloor, d-i} \subset \Lambda^{\mathrm{B}}$, $\mathfrak{s} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)}), \mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) \in M_{\lceil \frac{n}{2} \rceil, i}(\lambda^{(1)}) \times M_{\lfloor \frac{n}{2} \rfloor, d-i}(\lambda^{(2)}) \subset M^{\mathrm{B}}(\lambda)$, we have

$$\begin{aligned} C^\lambda_{\mathfrak{s},\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{t}} &= (C_{\lceil \frac{n}{2} \rceil, i})^{\lambda^{(1)}}_{\mathfrak{s}^{(1)},\mathfrak{s}^{(1)}} (C_{\lceil \frac{n}{2} \rceil, i})^{\lambda^{(1)}}_{\mathfrak{t}^{(1)},\mathfrak{t}^{(1)}} \otimes (C_{\lfloor \frac{n}{2} \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{s}^{(2)},\mathfrak{s}^{(2)}} (C_{\lfloor \frac{n}{2} \rfloor, d-i})^{\lambda^{(2)}}_{\mathfrak{t}^{(2)},\mathfrak{t}^{(2)}} \\ &\equiv \phi^{\mathrm{A}}_{\lambda^{(1)}}(C^{(1)}_{\mathfrak{s}}, C^{(1)}_{\mathfrak{t}}) \phi^{\mathrm{A}}_{\lambda^{(2)}}(C^{(2)}_{\mathfrak{s}}, C^{(2)}_{\mathfrak{t}}) C^\lambda_{\mathfrak{s},\mathfrak{t}} \mod S^{\mathrm{B}}_q(n,d)(< \lambda). \end{aligned}$$

□

Recall that in Proposition 2.6.1 we see that $S^{\mathrm{B}}_{Q,q}(2,1) \simeq \mathcal{H}^{\mathrm{A}}_{Q^{-1}}(\Sigma_2)$. In the following we show that the known cellular structure (due to Geck/Dipper-James) fails when $f^{\mathrm{B}} = Q^{-2} + 1$ is not invertible.

Example 6.1.2. Let $S^{\mathrm{B}}_{Q,q}(2,1) \simeq \mathcal{H}^{\mathrm{A}}_{Q^{-1}}(\Sigma_2) = K[t]/\langle t^2 - (Q^{-1} - Q)t + 1 \rangle$. We have

$$\Lambda = \left\{ \lambda = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array} \rhd \mu = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \right\}, \quad M(\lambda) = \{ \mathfrak{t} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \}, M(\mu) = \left\{ \mathfrak{s} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right\}.$$

The cellular basis elements are

$$C^\lambda_{\mathfrak{t}\mathfrak{t}} = \sum_{w \in \Sigma_2} Q^{-\ell(w)} T_w = 1 + Q^{-1} t, \quad C^\mu_{\mathfrak{s}\mathfrak{s}} = \sum_{w \in \Sigma_1 \times \Sigma_1} Q^{-\ell(w)} T_w = 1.$$

Firstly, we have $C^\mu_{\mathfrak{s}\mathfrak{s}} C^\mu_{\mathfrak{s}\mathfrak{s}} = 1 = C^\mu_{\mathfrak{s}\mathfrak{s}}$ and hence ϕ_μ is determined by $\phi_\mu(C_{\mathfrak{s}}, C_{\mathfrak{s}}) = 1$, which is nonzero. For λ we have

$$C^\lambda_{\mathfrak{t}\mathfrak{t}} C^\lambda_{\mathfrak{t}\mathfrak{t}} = 1 - Q^{-2} + (Q^{-2} + 1) Q^{-1} t \equiv (Q^{-2} + 1) C^\lambda_{\mathfrak{t}\mathfrak{t}} \mod A_{<\lambda}.$$

That is, ϕ_λ is determined by $\phi_\mu(C_{\mathfrak{t}}, C_{\mathfrak{t}}) = (Q^{-2} + 1)$, which can be zero when $f^{\mathrm{B}} = Q^{-2} + 1 = 0$. Therefore, $S^{\mathrm{B}}_{Q,q}(2,1)$ is not quasi-hereditary in an explicit way.

One can also see that $S^{\mathrm{B}}_{Q,q}(2,1)$ is not quasi-hereditary because if it were then it would have finite global dimension. However, $\mathcal{H}^{\mathrm{A}}_{Q^{-1}}(\Sigma_2)$ is a Frobenius algebra with infinite global dimension.

Conjecture 6.1.3. *The algebra $S^{\mathrm{B}}_{Q,q}(n,d)$ is quasi-hereditary if and only if $f^{\mathrm{B}}_d(Q,q)$ is invertible.*

7. Representation Type

7.1. Let A be a finite-dimensional algebra over a field K. A fundamental question one can ask about A is to describe its representation type. The algebra A is *semisimple* if and only if every finite-dimensional module (i.e., $M \in \mathrm{mod}(A)$) is a direct sum of simple modules. This means that indecomposable modules for A are simple. If A admits finitely many finite-dimensional indecomposable modules, A is said to be of *finite representation type*. If A does not have finite representation type A is of *infinite representation type*.

A deep theorem of Drozd states that finite dimensional algebras of infinite representation type can be split into two mutually exclusive categories: tame or wild. An algebra A has *tame representation type* if for each dimension there exists finitely many one-parameter families of indecomposable objects in $\mathrm{mod}(A)$. The indecomposable modules for algebras of tame representation type are classifiable. On the other hand, the algebras of *wild representation type* are those whose representation theory is as difficult to study as the representation theory of the free associative algebra $k\langle x,y\rangle$ on two variables. Classifying the finite-dimensional $k\langle x,y\rangle$-modules is very much an open question.

7.2. **Summary: Type A Results.** The following results from [EN01, Theorem 1.3(A)-(C)] summarize the representation type for the $\bar{q}$-Schur algebra for type A over K. Assume that $p = \mathrm{char}(K)$, $\bar{q} \in K^{\times}$ has multiplicative order l and $\bar{q} \neq 1$.

Theorem 7.2.1. *The algebra $S^{\mathrm{A}}_{\bar{q}}(n,r)$ is semisimple if and only if one of the following holds:*

(i) $n = 1$*;*
(i) $\bar{q}$ *is not a root of unity;*
(ii) $\bar{q}$ *is a primitive lth root of unity and $r < l$;*
(iii) $n = 2$*,* $p = 0$*,* $l = 2$ *and r is odd;*
(iv) $n = 2$*,* $p \geqslant 3$*,* $l = 2$ *and r is odd with $r < 2p + 1$.*

Theorem 7.2.2. *The algebra $S^{\mathrm{A}}_{\bar{q}}(n,r)$ has finite representation type but is not semi-simple if and only if $\bar{q}$ is a primitive lth root of unity with $l \leqslant r$, and one of the following holds:*

(i) $n \geqslant 3$ *and* $r < 2l$*;*
(ii) $n = 2$*,* $p \neq 0$*,* $l \geqslant 3$ *and* $r < lp$*;*
(iii) $n = 2$*,* $p = 0$ *and either* $l \geqslant 3$*, or* $l = 2$ *and r is even;*
(iv) $n = 2$*,* $p \geqslant 3$*,* $l = 2$ *and r even with $r < 2p$, or r is odd with $2p + 1 \leqslant r < 2p^2 + 1$.*

Theorem 7.2.3. *The algebra $S^{\mathrm{A}}_{\bar{q}}(n,r)$ has tame representation type if and only if $\bar{q}$ is a primitive lth root of unity and one of the following holds:*

(i) $n = 3$*,* $l = 3$*,* $p \neq 2$ *and* $r = 7, 8$*;*
(ii) $n = 3$*,* $l = 2$ *and* $r = 4, 5$*;*
(iii) $n = 4$*,* $l = 2$ *and* $r = 5$*;*
(iv) $n = 2$*,* $l \geqslant 3$*,* $p = 2$ *or* $p = 3$ *and* $pl \leqslant r < (p+1)l$*;*
(v) $n = 2$*,* $l = 2$*,* $p = 3$ *and* $r \in \{6, 19, 21, 23\}$*.*

7.3. In this section we summarize some of the fundamental results that are used to classify the representation type of Schur algebras. The first proposition can be verified by using the existence of the determinant representation for $S^{\mathrm{A}}_{\bar{q}}(n,r_1)$ (cf. [EN01, Proposition 2.4B]).

Proposition 7.3.1. *If $S^{\mathrm{A}}_{\bar{q}}(n,r_1) \otimes S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ has wild representation type then $S^{\mathrm{A}}_{\bar{q}}(n,r_1+n) \otimes S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ has wild representation type.*

Next we can present a sufficient criteria to show that the tensor product of type A Schur algebras has wild representation type.

Proposition 7.3.2. *Suppose that the Schur algebras $S^{\mathrm{A}}_{\bar{q}}(n,r_1)$ and $S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ are non-semisimple algebras. Then $S^{\mathrm{A}}_{\bar{q}}(n,r_1) \otimes S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ has wild representation type.*

Proof. First note that $S^{\mathrm{A}}_{\bar{q}}(n,r)$ is a quasi hereditary algebra and if $S^{\mathrm{A}}_{\bar{q}}(n,r)$ is not semisimple then it must have a block with at least two simple modules.

Suppose that S_1, S_2, S_3 are three simple modules in $S^{\mathrm{A}}_{\bar{q}}(n,r_1)$ with $\mathrm{Ext}^1_{S^{\mathrm{A}}_{\bar{q}}(n,r_1)}(S_1,S_2) \neq 0$ and $\mathrm{Ext}^1_{S^{\mathrm{A}}_{\bar{q}}(n,r_1)}(S_2,S_3) \neq 0$. Note that via the existence of the transposed duality,

$$\mathrm{Ext}^1_{S^{\mathrm{A}}_{\bar{q}}(n,r_1)}(S_i,S_j) \cong \mathrm{Ext}^1_{S^{\mathrm{A}}_{\bar{q}}(n,r_1)}(S_j,S_i)$$

for $i,j = 1,2,3$. Similarly, let T_1, T_2 be two simple modules for $S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ with $\mathrm{Ext}^1_{S^{\mathrm{A}}_{\bar{q}}(n,r_2)}(T_1,T_2) \neq 0$. Then the Ext^1-quiver for $S^{\mathrm{A}}_{\bar{q}}(n,r_1) \otimes S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ will have a subquiver of the form as in Figure 1 below. This quiver cannot be separated into a union of Dynkin diagrams or extended Dynkin diagrams. Consequently, $S^{\mathrm{A}}_{\bar{q}}(n,r_1) \otimes S^{\mathrm{A}}_{\bar{q}}(n,r_2)$ must has wild representation type.

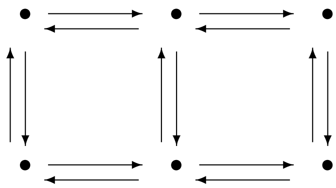

Figure 1.

The other case to consider is when the blocks of $S^{\mathrm{A}}_{\bar{q}}(n, r_1)$ and $S^{\mathrm{A}}_{\bar{q}}(n, r_2)$ have at most two simple modules. Let $\mathcal{B}_j$ be a block of $S^{\mathrm{A}}_{\bar{q}}(n, r_j)$ for $j = 1, 2$ with two simple modules. There are four simple modules in $\mathcal{B}_1 \otimes \mathcal{B}_2$ and the structure of the projective modules are the same as regular block for category $\mathcal{O}$ for the Lie algebra of type $\mathrm{A}_1 \times \mathrm{A}_1$ (cf. [FNP01, 4.2]). The argument in [FNP01, Lemma 4.2] can be use to show that $\mathcal{B}_1 \otimes \mathcal{B}_2$ has wild representation type. $\square$

7.4. The results in [EN01, Theorem 1.3(A)-(C)] entail using a different parameter $\bar{q}$ than the parameter q in our paper. The relationship is given by $\bar{q} = q^{-2}$ or equivalently $q^2 = (\bar{q})^{-1}$ with $S^{\mathrm{A}}_{q}(n, d) \cong S^{\mathrm{A}}_{\bar{q}}(n, d)$. This means that

- q is generic if and only if $\bar{q}$ is generic,
- q^2 is a primitive lth root of unity if and only if $\bar{q}$ is a primitive lth root of unity;
- if q is a primitive $2s$-th root of unity if and only if $\bar{q}$ is a primitive s-th root of unity;
- if q is a primitive $(2s+1)$-th root of unity if and only if $\bar{q}$ is a primitive $(2s+1)$-th root of unity.

Now let $n' \geqslant n$. By Proposition 4.2.1, under suitable conditions on n' and n, there exists an idempotent $e \in S^{\mathrm{B}}_{Q,q}(n', d)$ such that $S^{\mathrm{B}}_{Q,q}(n, d) \cong eS^{\mathrm{B}}_{Q,q}(n', d)e$. By using the proof in [EN01, Proposition 2.4B], one has the following result.

Proposition 7.4.1. *Let $n' \geqslant n$ with $n' \geqslant n$ and $n' \equiv n \mod 2$.*

(a) *If $S^{\mathrm{B}}_{Q,q}(n, d)$ is not semisimple then $S^{\mathrm{B}}_{Q,q}(n', d)$ is not semisimple.*
(b) *If $S^{\mathrm{B}}_{Q,q}(n, d)$ has wild representation type then $S^{\mathrm{B}}_{Q,q}(n', d)$ has wild representation type.*

7.5. **Type B Results.** Throughout this section, let $S^{\mathrm{B}}_{Q,q}(n, d)$ be the q-Schur algebra of Type B under the condition that the polynomial $f^{\mathrm{B}}_{d}(Q, q) \neq 0$. Moreover, assume that $q^2 \neq 1$ (i.e., $q \neq 1$ or a primitive 2nd root of unity). One can apply the isomorphism in Theorem 3.1.1 to determine the representation type for $S^{\mathrm{B}}_{Q,q}(n, d)$ from the Type A results stated in Section 7.2.

Theorem 7.5.1. *The algebra $S^{\mathrm{B}}_{Q,q}(n, d)$ is semisimple if and only if one of the following holds:*

(i) $n = 1$*;*
(ii) q *is not a root of unity;*
(iii) q^2 *is a primitive* l*th root of unity and* $d < l$*;*
(iv) $n = 2$ *and* d *arbitrary;*

Proof. The semisimplicity of (i)-(iii) follow by using Theorem 3.1.1 with Theorem 7.2.1. The semisimplicity of (iv) follows by Theorem 3.1.1 and the fact that $S^{\mathrm{A}}_{q}(1, d)$ is always semisimple.

Now assume that q^2 is a primitive lth root of unity, $d \geqslant l$, $n \geqslant 3$ and $l \geqslant 3$. Consider the case when $n = 3$. From Theorem 3.1.1,

$$S^{\mathrm{B}}_{Q,q}(3, d) \cong \bigoplus_{i=0}^{d} S^{\mathrm{A}}_{q}(2, i) \otimes S^{\mathrm{A}}_{q}(1, d-i). \tag{7.5.1}$$

If $d \geqslant l$ then $S^{\mathrm{A}}_{q}(2, l)$ appears as a summand of $S^{\mathrm{B}}_{Q,q}(3, d)$ (when $i = d - l$). For $l \geqslant 3$, $S^{\mathrm{A}}_{q}(2, l) \simeq S^{A}_{\bar{q}}(2, l)$ is not semisimple. It follows that $S^{\mathrm{B}}_{Q,q}(3, d)$ is not semisimple for $d \geqslant l$. One can repeat the same argument for $n = 4$ to show that $S^{\mathrm{B}}_{Q,q}(4, d)$ is not semisimple for $d \geqslant l$. Now apply Proposition 7.4.1(a) to deduce that $S^{\mathrm{B}}_{Q,q}(n, d)$ is not semisimple for $n \geqslant 3$ and $d \geqslant l$. □

Theorem 7.5.2. *The algebra $S^{\mathrm{B}}_{Q,q}(n, d)$ has finite representation type but is not semisimple if and only if q^2 is a primitive lth root of unity with $l \leqslant d$, and one of the following holds:*

(i) $n \geqslant 5$, $l \leqslant d < 2l$;
(ii) $n = 3$, $p = 0$ *and* $l \leqslant d$;
(iii) $n = 3$, $p \geqslant 2$ *and* $l \leqslant d < lp$;
(iv) $n = 4$, $p = 0$, $l = 2$ *and* $d \geqslant 4$ *with* d *odd.*
(v) $n = 4$, $p \geqslant 3$, $l = 2$ *and* $4 < d \leqslant 2p - 1$ *with* d *odd.*

The algebra $S^{\mathrm{B}}_{Q,q}(n, d)$ has tame representation type if and only if

(vi) $n = 3$, $l = 2$, $p = 3$ *and* $d = 6$;
(vii) $n = 3$, $l \geqslant 3$, $p = 2$ *or* 3 *and* $lp \leqslant d < l(p + 1)$;
(viii) $n = 4$, $l = 2$, $p = 3$ *and* $d = 7$.

Proof. We first reduce our analysis to the situation where $n = 3$ and 4. Assume that $n \geqslant 5$ so $\lceil \frac{n}{2} \rceil \geqslant 3$ and $\lfloor \frac{n}{2} \rfloor \geqslant 2$. By Theorem 7.2.1, the algebras $S^{\mathrm{A}}_q(2, l)$ and $S^{\mathrm{A}}_q(i, l+j)$ are not semisimple for $i \geqslant 3, j \geqslant 0$, and hence neither are $S^{\mathrm{A}}_q(\lceil \frac{n}{2} \rceil, l+j)$ and $S^{\mathrm{A}}_q(\lfloor \frac{n}{2} \rfloor, l)$ for $n \geqslant 5, j \geqslant 0$. Therefore,$S^{\mathrm{A}}_q(\lceil \frac{n}{2} \rceil, l+j) \otimes S^{\mathrm{A}}_q(\lfloor \frac{n}{2} \rfloor, l)$ has wild representation type by Proposition 7.3.2. It follows that $S^{\mathrm{B}}_{Q,q}(n, d)$ has wild representation type for $d \geqslant 2l, n \geqslant 5$. When $l \leqslant d < 2l$ and $n \geqslant 5$, one can use Theorem 3.1.1 in conjunction with Theorem 7.2.2 to prove that $S^{\mathrm{B}}_{Q,q}(n, d)$ has finite representation type.

Now consider the case when $n = 3$. The isomorphism (7.5.1) indicates that we can reduce our analysis to considering $S^{\mathrm{A}}_q(2, r)$. From this isomorphism and Theorem 7.2.2, one can verify that (i) when char $K = 0$ then $S^{\mathrm{B}}_{Q,q}(3, d)$ has finite representation type (but is not semisimple) for $l \leqslant d$, (ii) when char $K = p > 0$ then $S^{\mathrm{B}}_{Q,q}(3, d)$ has finite representation type (but is not semisimple) for $l \leqslant d < lp$, and (iii) when char $K = p > 0$, $S^{\mathrm{B}}_{Q,q}(3, d)$ has infinite representation type for $d \geqslant lp$.

For $n = 3$, one can also see that under conditions (vi) and (vii), $S^{\mathrm{B}}_{Q,q}(3, d)$ has tame representation type. Moreover, one can verify that $S^{\mathrm{B}}_{Q,q}(3, d)$ has wild representation type in the various complementary cases.

Finally let $n = 4$. From Proposition 7.3.2, $S^{\mathrm{A}}_q(2, l) \otimes S^{\mathrm{A}}_q(2, l)$ and $S^{\mathrm{A}}_q(2, l) \otimes S^{\mathrm{A}}_q(2, l + 1)$ has wild representation type for $l \geqslant 3$. Therefore, $S^{\mathrm{B}}_{Q,q}(4, d)$ has wild representation type for $d \geqslant 2l$ and $l \geqslant 3$. For $l = 2$, the same argument can be used to show that $S^{\mathrm{B}}_{Q,q}(4, d)$ has wild representation type for d-even and $d \geqslant 4$.

This reduces us to analyzing $S^{\mathrm{B}}_{Q,q}(4, d)$ when $l = 2$ and $d \geqslant 4$ is odd. By analyzing the components of $S^{\mathrm{B}}_{Q,q}(4, d)$ via the isomorphism in Theorem 3.1.1, one can show that for d odd: (i) $S^{\mathrm{B}}_{Q,q}(4, d)$ has finite representation type (not semisimple) for $4 \leqslant d \leqslant 2p - 1$ and $p \geqslant 3$, (ii) $S^{\mathrm{B}}_{Q,q}(4, d)$ has finite representation type (not semisimple) for $d \geqslant 4$ and $p = 0$, (iii) $S^{\mathrm{B}}_{Q,q}(4, d)$ has wild representation type for $d \geqslant 2p + 1$ for $p \geqslant 5$, and (iv) $S^{\mathrm{B}}_{Q,q}(4, d)$ has wild representation type for $d \geqslant 2p + 3$ for $p = 3$. One has then show that $S^{\mathrm{B}}_{Q,q}(4, 7)$ for $p = 3$, $l = 2$ has tame representation type since the component $S^{\mathrm{A}}_q(2, 6) \otimes S^{\mathrm{A}}_q(2, 1)$ has tame representation type and the remaining components have finite representation type.

□

Note that for the case $\bar{q} = 1$ (i.e., $q^2 = 1$) one obtains the classical Schur algebra for type A, one can use the results in [E93] [DN98] [DEMN99] to obtain classification results in this case for $S^{B}_{Q,q}(n, d)$.

8. Quasi-hereditary covers

In this section we first recall results on 1-faithful quasi-hereditary covers due to Rouquier [Ro08]. Then we demonstrate that our Schur algebra is a 1-faithful quasi-hereditary cover of the type B

Hecke algebra via Theorem 3.1.1. Hence, it module category identifies the category $\mathcal{O}$ for the rational Cherednik algebra of type B, see Theorem 8.3.3. A comparison of our Schur algebra with Rouquier's Schur-type algebra is also provided.

8.1. **1-faithful covers.** Let $\mathcal{C}$ be a category equivalent to the module category of a finite dimensional projective K-algebra A, and let $\Delta = \{\Delta(\lambda)\}_{\lambda \in \Lambda}$ be a set of objects of $\mathcal{C}$ indexed by an interval-finite poset structure Λ. Following [Ro08], we say that $\mathcal{C}$ (or $(\mathcal{C}, \Delta)$) is a *highest weight category* if the following conditions are satisfied:

(H1) $\mathrm{End}_{\mathcal{C}}(\Delta(\lambda)) = K$ for all $\lambda \in \Lambda$;
(H2) If $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) \neq 0$ then $\lambda \leqslant \mu$;
(H3) If $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$ then $M = 0$;
(H4) For each $\Delta(\lambda) \in \Delta$ there is a projective module $P(\lambda) \in \mathcal{C}$ such that $\ker(P(\lambda) \to \Delta(\lambda))$ has a Δ-filtration, i.e., finite filtrations whose quotients are isomorphic to objects in Δ.

Let A-mod be the category of finitely generated A-modules. The algebra A is called a *quasi-hereditary cover* of B if the conditions below hold:

(C1) A-mod admits a highest weight category structure $(A\text{-mod}, \Delta)$;
(C2) $B = \mathrm{End}_A(P)$ for some projective $P \in A$-mod;
(C3) The restriction of $F = \mathrm{Hom}_A(P, -)$ to the category of finitely generated projective A-modules is fully faithful.

Quasi-hereditary covers are sometimes called highest weight covers since the notion of highest weight category corresponds to that of split quasi-hereditary algebras [Ro08, Theorem 4.16]. We also say that (A, F) is a quasi-hereditary cover of B. Moreover, a category $\mathcal{C}$ (or the pair $(\mathcal{C}, F)$) is said to be a quasi-hereditary cover of B if $\mathcal{C} \simeq A$-mod for some quasi-hereditary cover (A, F) of B.

Following [Ro08], a quasi-hereditary cover A of B is *i-faithful* if

$$\mathrm{Ext}^j_A(M, N) \simeq \mathrm{Ext}^j_B(FM, FN) \quad \text{for} \quad j \leqslant i, \tag{8.1.1}$$

and for all $M, N \in A$-mod admitting Δ-filtrations. Furthermore, a quasi-hereditary cover $(\mathcal{C}, F)$ of B is said to be i-faithful if the diagram below commutes for some quasi-hereditary cover (A, F') of B:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\simeq"] \arrow[dr, "F"'] & & A\text{-mod} \arrow[dl, "F'"] \\ & B\text{-mod} & \end{tikzcd}$$

Rouquier proved in [Ro08, Theorem 4.49] a uniqueness theorem for the 1-faithful quasi-hereditary covers which we paraphrase below:

Proposition 8.1.1. *Let B be a finite projective K-algebra that is split semisimple, and let $(\mathcal{C}_i, F_i)$ for $i = 1, 2$ be 1-faithful quasi-hereditary covers of B with respect to the partial order $\leqslant_i$ on $\mathrm{Irr}(B)$. If $\leqslant_1$ is a refinement of $\leqslant_2$ then there is an equivalence $\mathcal{C}_1 \simeq \mathcal{C}_2$ of quasi-hereditary covers of B inducing the bijection $\mathrm{Irr}(\mathcal{C}_1) \simeq \mathrm{Irr}(B) \simeq \mathrm{Irr}(\mathcal{C}_2)$.*

8.2. **Rational Cherednik algebras.** Let (W, S) be a finite Coxeter group, and let A_W be the corresponding rational Cherednik algebra over $\mathbb{C}[h_u; u \in U]$ as in [Ro08], where $U = \bigsqcup_{s \in S} \{s\} \times \{1, \dots, e_s\}$ and e_s is the size of the pointwise stabilizer in W of the hyperplane corresponding to s. If $W = W^{\mathrm{B}}(d)$ and $S = \{s_0, s_1\}$ then $U = \{(s_i, j) \mid 0 \leqslant i, j \leqslant 1\}$. In this case we assume that

$$h_{(s_1,0)} = h, \quad h_{(s_1,1)} = 0, \quad h_{(s_0,i)} = h_i \quad \text{for} \quad i = 0, 1. \tag{8.2.1}$$

Remark 8.2.1. In [EG02] the rational Cherednik algebra $\mathbf{H}_{t,c}$ is defined for a parameter $t \in \mathbb{C}$, and a W-equivariant map $c : S \to \mathbb{C}$. The two algebras, A_W and $\mathbf{H}_{t,c}$, coincide if $t = 1$, $h_{(s,0)} = 0$ and $h_{(s,1)} = c(s)$ for all $s \in S$.

Let $\mathcal{O}_W$ be the category of finitely generated A_W-modules that are locally nilpotent for $S(V)$. It is proved in [GGOR03] that $(\mathcal{O}_W, \Delta_W)$ is a highest weight category of $\mathcal{H}(W)$-mod

$$\Delta_W = \{\Delta(E) := A_W \otimes_{S(V)\rtimes W} E \mid E \in \mathrm{Irr}(W)\},$$

See [Ro08, 3.2.1–3] for the partial order $\leqslant$ on $\mathrm{Irr}(W)$. Let $\Lambda_2^+(d)$ be the poset of all bipartitions of d on which the dominance order $\trianglelefteq$ is given by $\lambda \trianglelefteq \mu$ if, for all $s \geqslant 0$,

$$\sum_{j=1}^{s} |\lambda_j^{(1)}| \leqslant \sum_{j=1}^{s} |\mu_j^{(1)}|, \quad |\lambda^{(1)}| + \sum_{j=1}^{s} |\lambda_j^{(r)}| \leqslant |\mu^{(1)}| + \sum_{j=1}^{s} |\mu_j^{(r)}|. \tag{8.2.2}$$

For $\lambda \in \Lambda_2^+(d)$, set

$$W_\lambda^{\mathrm{B}}(d) = C_2^d \rtimes (\Sigma_{\lambda^{(1)}} \times \Sigma_{\lambda^{(2)}}), \tag{8.2.3}$$

Set

$$I_\lambda(1) = \{1, \dots, |\lambda^{(1)}|\}, \quad I_\lambda(2) = \{|\lambda^{(1)}| + 1, \dots, d\}. \tag{8.2.4}$$

Following [Ro08, 6.1.1], there is a bijection

$$\Lambda_2^+(d) \to \mathrm{Irr}(W^{\mathrm{B}}(d)), \quad \lambda = (\lambda^{(1)}, \lambda^{(2)}) \mapsto \chi_\lambda = \mathrm{Ind}_{W_\lambda^{\mathrm{B}}(d)}^{W^{\mathrm{B}}(d)}(\chi_{\lambda^{(1)}} \otimes \phi^{(2)}\chi_{\lambda^{(2)}}), \tag{8.2.5}$$

where χ_λ is the irreducible character of $W^{\mathrm{B}}(d)$ corresponding to λ, and $\phi^{(2)}$ is the 1-dimensional character of $C_2^{I_\lambda(2)} \rtimes \Sigma_{I_\lambda(2)}$ whose restriction to $C_2^{I_\lambda(2)}$ is det and the restriction to $\Sigma_{I_\lambda(2)}$ is trivial.

Rouquier showed that the order $\leqslant$ is a refinement of the dominance order $\trianglelefteq$ under an assumption on the parameters h, h_i's for the rational Cherednik algebra as follows:

Lemma 8.2.2. [Ro08, Proposition 6.4] *Assume that $W = W^B(d)$, $h \leqslant 0$ and $h_1 - h_0 \geqslant (1-d)h$ (see (8.2.1)). Let $\lambda, \mu \in \Lambda_2^+(d)$. If $\lambda \trianglelefteq \mu$, then $\chi_\lambda \leqslant \chi_\mu$ on $\mathrm{Irr}(W)$.*

Remark 8.2.3. The assumption in Lemma 8.2.2 on the parameters is equivalent to $c(s_0) = h_1 \geqslant 0$ using Etingof-Ginzburg's convention.

Let KZ_W be the KZ functor $\mathcal{O}_W \to \mathcal{H}(W)$-mod. We paraphrase [Ro08, Theorem 5.3] in our setting as below:

Proposition 8.2.4. *If $W = W^{\mathrm{B}}(d)$ and $\mathcal{H}(W) = \mathcal{H}_{Q,q}^{\mathrm{B}}(d)$, then $(\mathcal{O}_W, KZ_W)$ is a quasi-hereditary cover of $\mathcal{H}(W)$-mod. Moreover, the cover is 1-faithful if $(q^2+1)(Q^2+1) \neq 0$.*

It is shown in [Ro08] that under suitable assumptions, $\mathcal{O}_{W^{\mathrm{B}}(d)}$ is equivalent to the module category of a Schur-type algebra $S^{\mathrm{R}}(d)$ which does not depend on n using the uniqueness property Proposition 8.1.1. Below we give an interpretation in our setting.

Let $\Lambda_2(d)$ be the set of all bicompositions of d. In [DJM98] a cyclotomic Schur algebra over $\mathbb{Q}(q, Q, Q_1, Q_2)$ for each saturated subset $\Lambda \subset \Lambda_2(d)$, which specializes to cyclotomic Schur algebras $S_Q(\Lambda)$ over K is defined (see Section 9.2). Moreover, in [Ro08] an algebra $S_Q(\Lambda)$ is defined that is Morita equivalent to $S_Q(\Lambda)$ as given below:

$$S^{\mathrm{R}}(d) := \mathrm{End}_{\mathcal{H}_{Q,q}^{\mathrm{B}}(d)}(P_d), \quad P_d := \bigoplus_{\lambda \in \Lambda_2^+(d)} m_\lambda \mathcal{H}_{Q,q}^{\mathrm{B}}(d). \tag{8.2.6}$$

where m_λ is defined in (9.2.11). Note that $S^{\mathrm{R}}(d)$ does not depend on n. Set

$$F_d^{\mathrm{R}} = \mathrm{Hom}_{S^{\mathrm{R}}(d)}(P_d, -) : S^{\mathrm{R}}(d)\text{-mod} \to \mathcal{H}_{Q,q}^{\mathrm{B}}(d)\text{-mod}. \tag{8.2.7}$$

Proposition 8.2.5. [Ro08, Theorem 6.6]

(a) The category $\mathrm{Mod}(S^{\mathrm{R}}(d))$ is a highest weight category for the dominance order;
(b) $(S^{\mathrm{R}}(d), F_d^{\mathrm{R}})$ is a quasi-hereditary cover of $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$;

(c) The cover $(S^{\mathrm{R}}(d), F_d^{\mathrm{R}})$ is 1-faithful if

$$(q^2+1)(Q^2+1) \neq 0, \quad \text{and} \quad f_{Q,q}^{\mathrm{B}}(d) \cdot \prod_{i=1}^{d}(1+q^2+\cdots+q^{2(i-1)}) \neq 0. \tag{8.2.8}$$

The category $\mathcal{O}$ for the type B rational Cherednik algebra together with its KZ functor can then be identified by combining Propositions 8.1.1, 8.2.4 and 8.2.5. In other words, the following diagram commutes if (8.2.8) holds:

$$\begin{tikzcd} \mathcal{O}_{W^{\mathrm{B}}(d)} \arrow[rr, "\simeq"] \arrow[dr, "KZ_{W^{\mathrm{B}}(d)}"'] & & S^{\mathrm{R}}(d)\text{-mod} \arrow[dl, "F_d^{\mathrm{R}}"] \\ & \mathcal{H}_{Q,q}^{\mathrm{B}}(d)\text{-mod} & \end{tikzcd}$$

8.3. **1-faithfulness of $S_{Q,q}^{\mathrm{B}}(n,d)$-mod.** Let ℓ be the multiplicative order of q^2 in $K^\times$. In this section we use the following assumptions:

$$f_d^{\mathrm{B}}(Q,q) = \prod_{i=1-d}^{d-1}(Q^{-2}+q^{2i}) \in K^\times, \quad r := \lfloor \tfrac{n}{2} \rfloor \geqslant d, \quad \ell \geqslant 4. \tag{8.3.1}$$

As a consequence, there exists a type B Schur functor by Proposition 4.1.1. For type A, it is known in [HN04] that the q-Schur algebra is a 1-faithful quasi-hereditary cover of the type A Hecke algebra if $\ell \geqslant 4$. Moreover, Theorem 3.1.1 applies and hence we will see shortly that $S_{Q,q}^{\mathrm{B}}(n,d)$ is a 1-faithful quasi-hereditary cover of $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$. Furthermore, Proposition 8.1.1 implies that we have a concrete realization for the category $\mathcal{O}$ for the type B rational Cherednik algebra together with its KZ functor using our Schur algebra.

Corollary 8.3.1. *If $f_d^{\mathrm{B}} \in K^\times$, then $S_{Q,q}^{\mathrm{B}}(n,d)$-mod is a highest weight category.*

Proof. It follows immediately from the isomorphism with the direct sum of type A q-Schur algebras that $S_{Q,q}^{\mathrm{B}}(n,d)$-mod is a highest weight category. □

In below we characterize a partial order for highest weight category $S_{Q,q}^{\mathrm{B}}(n,d)$-mod obtained via Corollary 8.3.1 and the dominance order for type A. Denote the set of all N-step partitions of D by $\Lambda^{\mathrm{A}}(N,D)$. Set

$$\Delta_{N,D}^{\mathrm{A}} = \{\Delta^{\mathrm{A}}(\lambda) \mid \lambda \in \Lambda^{\mathrm{A}}(N,D)\}. \tag{8.3.2}$$

Now $\Delta_{N,D}^{\mathrm{A}}$ is a poset with respect to the dominance order $\trianglelefteq$ on $\Lambda^{\mathrm{A}}(N,D)$. It is well known that for all non-negative integers N and D, $(S_q^{\mathrm{A}}(N,D)\text{-mod}, \Delta_{N,D}^{\mathrm{A}})$ is a highest weight category.

Recall $\mathcal{F}_S$ from (4.1.11) and $\Lambda^{\mathrm{B}}(n,d)$ from (5.2.1). Set

$$\Delta_{n,d}^{\mathrm{B}} = \{\Delta^{\mathrm{B}}(\lambda) := \mathcal{F}^{-1}(\Delta^{\mathrm{A}}(\lambda^{(1)}) \otimes \Delta^{\mathrm{A}}(\lambda^{(2)})) \mid \lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^{B}(n,d)\}. \tag{8.3.3}$$

Now $\Delta_{n,d}^{\mathrm{B}}$ is a poset with respect to the dominance order (also denoted by $\trianglelefteq$) on $\Lambda^{B}(n,d) \subset \Lambda_2^+(d)$. Hence, $(S_q^{\mathrm{B}}(n,d)\text{-mod}, \trianglelefteq)$ is a highest weight category.

Lemma 8.3.2. *Assume that $S_{Q,q}^{\mathrm{B}}(n,d)$ is a quasi-hereditary cover of $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$. If* (8.3.1) *holds, then the cover is 1-faithful.*

Proof. Write $A = S_{Q,q}^{\mathrm{B}}(n,d), B = \mathcal{H}_{Q,q}^{\mathrm{B}}(d), S' = S_q^{\mathrm{A}}(\lceil \frac{n}{2} \rceil, i), S'' = S_q^{\mathrm{A}}(\lfloor \frac{n}{2} \rfloor, d-i)$ for short. We need to show that, for all M, N admitting Δ^{B}-filtrations,

$$\mathrm{Ext}_A^i(M,N) \simeq \mathrm{Ext}_{eAe}^i(F_{n,d}^{\mathrm{B}}M, F_{n,d}^{\mathrm{B}}N), \quad i \leqslant 1.$$

Recall $\mathcal{F}_S$ from (4.1.11). Write $\mathcal{F}_S M = \bigoplus_i M_i' \otimes M_i''$ and $\mathcal{F}_S N = \bigoplus_i N_i' \otimes N_i''$ for some $M_i', N_i' \in \mathrm{Mod}(S')$ and $M_i'', N_i'' \in \mathrm{Mod}(S'')$. From construction we see that all M_i', M_i'', N_i', N_i'' admit Δ^{A}-filtrations since M, N have Δ^{B}-filtrations.

For $\lfloor\frac{n}{2}\rfloor \geqslant d \geqslant i \geqslant 0$, we abbreviate the type A Schur functors (see (4.1.1)) by $F' = F^{\mathrm{A}}_{\lceil\frac{n}{2}\rceil,i}$, $F'' = F^{\mathrm{A}}_{\lfloor\frac{n}{2}\rfloor,d-i}$. Since the type A q-Schur algebras are 1-faithful provided $\ell \geqslant 4$, for $j \leqslant 1$ we have

$$\begin{aligned}\mathrm{Ext}^j_{S'}(M'_i, N'_i) &\simeq \mathrm{Ext}^j_{\mathcal{H}_q(\Sigma_{i+1})}(F'M'_i, F'N'_i),\\ \mathrm{Ext}^j_{S''}(M''_i, N''_i) &\simeq \mathrm{Ext}^j_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M''_i, F''N''_i).\end{aligned} \tag{8.3.4}$$

We show first it is 0-faithful. We have

$$\begin{aligned}\mathrm{Hom}_A(M, N) &\simeq \mathrm{Hom}_{\bigoplus_{i=0}^d S'\otimes S''}\Big(\mathcal{F}_S M, \mathcal{F}_S N\Big)\\ &\simeq \bigoplus_{i=0}^d \mathrm{Hom}_{S'}(M'_i, N'_i) \otimes \mathrm{Hom}_{S''}(M''_i, N''_i)\\ &\simeq \bigoplus_{i=0}^d \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})}(F'M'_i, F'N'_i) \otimes \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M''_i, F''N''_i)\\ &\simeq \bigoplus_{i=0}^d \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})\otimes\mathcal{H}_q(\Sigma_{d-i+1})}\Big(F'M'_i \otimes F'M''_i, F''N'_i \otimes F''N''_i\Big)\\ &\simeq \bigoplus_{i=0}^d \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})\otimes\mathcal{H}_q(\Sigma_{d-i+1})}(\mathcal{F}_H F^\flat_{n,d} M, \mathcal{F}_H F^\flat_{n,d} N)\\ &\simeq \mathrm{Hom}_B(F^\flat_{n,d} M, F^\flat_{n,d} N).\end{aligned} \tag{8.3.5}$$

Note that the second last isomorphism follows from Proposition 4.1.2. For 1-faithfulness, we have

$$\begin{aligned}\mathrm{Ext}^1_A(M, N) &\simeq \bigoplus_{i=0}^d \big((\mathrm{Ext}^1_{S'}(M'_i, N'_i) \otimes \mathrm{Hom}_{S''}(M''_i, N''_i))\\ &\qquad \oplus (\mathrm{Hom}_{S'}(M'_i, N'_i) \otimes \mathrm{Ext}^1_{S''}(M''_i, N''_i))\big)\\ &\simeq \bigoplus_{i=0}^d \big((\mathrm{Ext}^1_{\mathcal{H}_q(\Sigma_{i+1})}(F'M'_i, F'N'_i) \otimes \mathrm{Hom}_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M''_i, F''N''_i))\\ &\qquad \oplus (\mathrm{Hom}_{\mathcal{H}_q(\Sigma_{i+1})}(F'M'_i, F'N'_i) \otimes \mathrm{Ext}^1_{\mathcal{H}_q(\Sigma_{d-i+1})}(F''M''_i, F''N''_i))\\ &\simeq \bigoplus_{i=0}^d \mathrm{Ext}^1_{\mathcal{H}_q(\Sigma_{i+1})\otimes\mathcal{H}_q(\Sigma_{d-i+1})}(\mathcal{F}_H F^\flat_{n,d} M, \mathcal{F}_H F^\flat_{n,d} N)\\ &\simeq \mathrm{Ext}^1_B(F^\flat_{n,d} M, F^\flat_{n,d} N).\end{aligned} \tag{8.3.6}$$

□

Theorem 8.3.3. *Assume that $W = W^B(d)$, $h \leqslant 0$, $h_1 - h_0 \geqslant (1-d)h$ (see (8.2.1)) and $(q^2+1)(Q^2+1) \in K^\times$. If (8.3.1) holds, then there is an equivalence $\mathcal{O}_W \simeq S^{\mathrm{B}}_{Q,q}(n,d)$-mod of quasi-hereditary covers. In other words, the following diagram commutes:*

$$\begin{tikzcd} \mathcal{O}_W \arrow[rr, "\simeq"] \arrow[dr, "KZ_W"'] & & S^{\mathrm{B}}_{Q,q}(n,d)\text{-mod} \arrow[dl, "F^\flat_{n,d}"] \\ & \mathcal{H}^{\mathrm{B}}_{Q,q}(d)\text{-mod} & \end{tikzcd}$$

Proof. The theorem follows by combining Propositions 8.1.1, 8.2.4 and Lemmas 8.2.2, 8.3.2. □

Remark 8.3.4. The uniqueness theorem for 1-faithful quasi-hereditary covers also applies on our Schur algberas and Rouquier's Schur-type algebras. That is, the following diagram commutes provided

(8.2.8) and (8.3.1) hold:

$$\begin{tikzcd}
S^{\mathrm{R}}(d)\text{-mod} \arrow[rr, "\simeq"] \arrow[dr, "F^{\mathrm{R}}_d"'] & & S^{\mathrm{B}}_{Q,q}(n,d)\text{-mod} \arrow[dl, "F^{\flat}_{n,d}"] \\
& \mathcal{H}^{\mathrm{B}}_{Q,q}(d)\text{-mod} &
\end{tikzcd}$$

9. Variants of q-Schur algebras of type B/C

It is interesting that the type A q-Schur algebra admits quite a few distinct generalizations in type B/C in literature. This is due to that the type A q-Schur algebra can be realized differently due to the following realizations of the tensor space $(K^n)^{\otimes d}$: (1) A combinatorial realization as a quantized permutation module (cf. [DJ89]); (2) A geometric realization as the convolution algebra on GL_n-invariant pairs consisting of a n-step partial flag and a complete flag over finite field (cf. [BLM90]).

In the following sections we provide a list of q-Schur duality/algebras of type B/C in literature, paraphrased so that they are all over K, and with only one parameter q. These algebras are all of the form $\mathrm{End}_{\mathcal{H}^{\mathrm{B}}_q(d)}(V^{\otimes d})$ for some tensor space that may have a realization $V^{\otimes d} \simeq \bigoplus_{\lambda\in I} M^\lambda$ via induced modules. Considering the specialization at $q = 1$, we have

$$M^\lambda\Big|_{q=1} = \mathrm{ind}_{H_\lambda}^{W^{\mathrm{B}}(d)} U, \quad H_\lambda \leqslant W^{\mathrm{B}}(d) \text{ is a subgroup}, \quad U \text{ is usually the trivial module.}$$

We summarize the properties of the q-Schur algebras in the following table:

	Coideal q-Schur Algebra $S^{\mathrm{B}}_q(n,d)$	Cyclotomic Schur Algebra $S_q(\Lambda)$	Sakamoto-Shoji Algebra $S^{\mathrm{B}}_q(a,b,d)$
Index set I	compositions $\lambda = (\lambda_i)_{i\in I(n)}$ with constraints on λ_i	bicompositions $\lambda = (\lambda^{(1)}, \lambda^{(2)})$	unclear
Subgroup H_λ	$W^{\mathrm{B}}(\lambda_0) \times \Sigma_{(\lambda_1,\dots\lambda_r)}$	$(C_2^{\lvert\lambda^{(1)}\rvert} \times C_2^{\lvert\lambda^{(2)}\rvert}) \rtimes \Sigma_\lambda$	unknown
Module U	trivial	nontrivial	
Schur duality	$(U^{\mathrm{B}}_q(n), \mathcal{H}^{\mathrm{B}}_q(d))$	unknown	$(U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b), \mathcal{H}^{\mathrm{B}}_q(d))$
Cellularity	new [LNX]	known [DJM98]	unknown
Quasi-heredity	new [LNX]	known [DJM98]	unknown
Schur functor	new [LNX]	unknown	unknown

For completeness a more involved q-Schur algebra (referred as the q-Schur^2 algebras) of type B is studied in [DS00]. We also distinguish the coideal q-Schur algebras from the slim cyclotomic Schur algebras constructed in [DDY18].

9.1. **The coideal Schur algebra $S^{\mathrm{B}}_q(n,d)$.** This is the main object in this paper which we have been calling the q-Schur algebra of type B. To distinguish it from the other variants we call them for now the coideal Schur algebras since they are homomorphic images of coideal subalgebras.

For the equal-parameter case, a geometric Schur duality is established between $\mathcal{H}^{\mathrm{B}}_q(d)$ and the coideal subalgebra $U^{\mathrm{B}}_q(n)$ as below (cf. [BKLW18]):

$$\begin{tikzcd}
U^{\mathrm{B}}_q(n) \arrow[d] & & & & \\
S^{\mathrm{B}}_q(n,d) \arrow[r, phantom, "\curvearrowright"] & T^{\mathrm{B}}_{\mathrm{geo}}(n,d) \simeq (K^n)^{\otimes d} \simeq T^{\mathrm{B}}_{\mathrm{alg}}(n,d) & \arrow[l, phantom, "\curvearrowleft"] \mathcal{H}^{\mathrm{B}}_q(d)
\end{tikzcd}$$

Note that a construction using type C flags is also available, and it produces isomorphic Schur algebras and hence coideals. A combinatorial realization $T^{\mathrm{B}}_{\mathrm{alg}}(n,d)$ as a quantized permutation module is also available along the line of Dipper-James.

For the case with two parameters, the algebra $S^{\mathrm{B}}_{Q,q}(n,d)$, when n is even, was first introduced by Green and it is called the hyperoctahedral q-Schur algebra [Gr97]. A two-parameter upgrade for the picture above is partially available – a Schur duality is obtained in [BWW18] between the two-parameter Hecke algebra $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ and the two-parameter coideal $\mathbb{U}^{\mathrm{B}}_n$ over the tensor space $\mathbb{Q}(Q,q)$; a two-parameter upgrade for $T^{\mathrm{B}}_{\mathrm{alg}}(n,d)$ is studied in [LL18] – while a two-parameter upgrade for $T^{\mathrm{B}}_{\mathrm{geo}}(n,d)$ remains unknown since dimension counting over finite fields does not generalize to two parameters naively.

To our knowledge, this is the only q-Schur algebras for the Hecke algebras of type B that admit a coordinate algebra type construction and a notion of the Schur functors with the existence of appropriate idempotents.

9.2. **Cyclotomic Schur algebras.** The readers will be reminded shortly that the cyclotomic Hecke algebra $\mathbb{H}(r,1,d)$ of type $G(r,1,d)$ is isomorphic to $\mathcal{H}^{\mathrm{B}}_q(d)$ at certain specialization when $r=2$. For each saturated subset Λ of the set of all bicompositions, Dipper-James-Mathas (cf. [DJM98]) define the cyclotomic Schur algebra $\mathbb{S}(\Lambda)$:

$$S_q(\Lambda) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_q(d)} T(\Lambda),$$

where $T(\Lambda)$ is a quantized permutation module that has no known identification with a tensor space. While a cellular structure (and hence a quasi-heredity) is obtained for $S_q(\Lambda)$, it is unclear if it has a double centralizer property. We also remark that there is no known identification of $T^{\mathrm{B}}_{\mathrm{alg}}(n,d)$ with a $T(\Lambda)$ for some Λ.

Let $R=\mathbb{Q}(q,Q,Q_1,Q_2)$. The cyclotomic Hecke algebra (or Ariki-Koike algebra) $\mathbb{H}=\mathbb{H}(2,1,d)$ is the R-algebra generated by $T^\Delta_0,\dots,T^\Delta_{d-1}$ subject to the relations below, for $1\leqslant i\leqslant d-1, 0\leqslant j< k-1\leqslant d-2$:

$$(T^\Delta_0 - Q_1)(T^\Delta_0 - Q_2) = 0, \quad (T^\Delta_i + 1)(T^\Delta_0 - q_\Delta) = 0, \tag{9.2.1}$$

$$(T^\Delta_0 T^\Delta_1)^2 = (T^\Delta_1 T^\Delta_0)^2, \quad T^\Delta_i T^\Delta_{i+1} T^\Delta_i = T^\Delta_{i+1} T^\Delta_i T^\Delta_{i+1}, \quad T^\Delta_k T^\Delta_j = T^\Delta_j T^\Delta_k. \tag{9.2.2}$$

Next we rewrite the setup in . *loc. cit.* using the following identifications:

$$q_\Delta \leftrightarrow q^{-2}, \quad T^\Delta_i \leftrightarrow q^{-1} T_i. \tag{9.2.3}$$

Under the identification, the Jucy-Murphy elements are, for $m\geqslant 1$,

$$\begin{aligned} L_m &= (q_\Delta)^{1-m} T^\Delta_{m-1} \dots T^\Delta_0 \dots T^\Delta_{m-1} \\ &= (qT^\Delta_{m-1}) \dots (qT^\Delta_0) \dots (qT^\Delta_{m-1}) \\ &= T_{m-1} \dots T_0 \dots T_{m-1}. \end{aligned} \tag{9.2.4}$$

Then the cyclotomic relation is

$$(q^{-1}T_0 - Q_1)(q^{-1}T_0 - Q_2) = 0, \quad \text{or} \quad (T_0 - qQ_1)(T_0 - qQ_2) = 0. \tag{9.2.5}$$

This is equivalent to our Hecke relation at the specialization below:

$$Q_1 = -q^{-1}Q, \quad Q_2 = q^{-1}Q^{-1}. \tag{9.2.6}$$

In summary we have the following isomorphism of K-algebras.

Proposition 9.2.1. *The type B Hecke algebra $\mathcal{H}^{\mathrm{B}}_{Q,q}(d)$ is isomorphic to the cyclotomic Hecke algebra $\mathbb{H}(2,1,d)$ at the specialization $Q_1=-q^{-1}Q, Q_2=q^{-1}Q^{-1}$.*

For a composition $\lambda=(\lambda_1,\dots,\lambda_\ell)\in\mathbb{N}^\ell$ of ℓ parts write

$$|\lambda| = \lambda_1 + \ldots + \lambda_\ell, \quad \text{and} \quad \ell(\lambda) = \ell. \tag{9.2.7}$$

A bicomposition of d is a pair $\lambda=(\lambda^{(1)},\lambda^{(2)})$ of compositions such that $|\lambda^{(1)}|+|\lambda^{(2)}|=d$. We denote the set of bicompositions of d by $\Lambda_2=\Lambda_2(d)$. A bicomposition λ is a bipartition if $\lambda^{(1)},\lambda^{(2)}$ are both partitions. The set of bipartitions of d is denoted by $\Lambda^+_2=\Lambda^+_2(d)$.

Following [DJM98], the cyclotomic Schur algebras can be defined for any saturated subset Λ of the set $\Lambda_2(d)$ of all bicompositions of d. That is, any subset Λ of Λ_2 satisfying the condition below:

$$\text{If } \mu \in \Lambda, \nu \in \Lambda_2^+(d) \text{ and } \nu \rhd \mu, \text{ then } \nu \in \Lambda. \tag{9.2.8}$$

For each Λ we define a cyclotomic Schur algebra $\mathbb{S}(\Lambda) = \mathrm{End}_{\mathbb{H}}\left(\bigoplus_{\lambda\in\Lambda} m_\lambda \mathbb{H}\right)$, where

$$m_\lambda = u^+_{\ell(\lambda^{(1)})} x_\lambda, \quad u^+_{\ell(\lambda^{(1)})} = \prod_{m=1}^{\ell(\lambda^{(1)})} (L_m - Q_2), \quad x_\lambda = \sum_{w \in \Sigma_\lambda} T_w, \tag{9.2.9}$$

and $\Sigma_\lambda = \Sigma_\lambda^{(1)} \times \Sigma_\lambda^{(2)}$ is the Young subgroup of Σ_d. The specialization $S_Q(\Lambda)$ of $\mathbb{S}(\Lambda)$ at $Q_1 = -q^{-1}Q, Q_2 = q^{-1}Q^{-1}$ is then given by

$$S_Q(\Lambda) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_{Q,q}}\left(\bigoplus_{\lambda\in\Lambda} m_\lambda \mathcal{H}^{\mathrm{B}}_{Q,q}\right), \tag{9.2.10}$$

where

$$m_\lambda = (L_1 - q^{-1}Q^{-1}) \dots (L_{\ell(\lambda^{(1)})} - q^{-1}Q^{-1}) x_\lambda. \tag{9.2.11}$$

Let $\mathcal{T}_0(\lambda, \mu)$ be the set of semi-standard λ-tableaux of type μ, that is, any $T = (T^{(1)}, T^{(2)}) \in \mathcal{T}_0(\lambda, \mu)$ satisfies the conditions below:

(S0) T is a λ-tableau whose entries are ordered pairs (i, k), and the number of (i, j)'s appearing is equal to $\mu_i^{(k)}$;
(S1) entries in each row of each component $T^{(k)}$ are non-decreasing;
(S2) entries in each column of each component $T^{(k)}$ are strictly increasing;
(S3) entries in $T^{(2)}$ must be of the form $(i, 2)$.

We note that the dimension of the cyclotomic Schur algebra Λ is given by

$$\dim S_Q(\Lambda) = \sum_{\substack{\lambda \in \Lambda_2^+(d) \\ \mu,\nu\in\Lambda}} |\mathcal{T}_0(\lambda, \mu)| \cdot |\mathcal{T}_0(\lambda, \nu)|. \tag{9.2.12}$$

It is then define a "tensor space" $T_Q(\Lambda) = \bigoplus_{\lambda\in\Lambda} m_\lambda \mathcal{H}^{\mathrm{B}}_{Q,q}$ which has an obvious $S_q(\Lambda)$-$\mathcal{H}^{\mathrm{B}}_q(d)$-bimodule structure.

Example 9.2.2. Let

$$\Lambda_{a,b} = \Lambda_{a,b}(d) = \{\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2(d) \mid \ell(\lambda^{(1)}) \leqslant a, \ell(\lambda^{(2)}) \leqslant b\}. \tag{9.2.13}$$

Recall that the dominance partial order in $\Lambda_2^+(1)$ is given by $\mu_2 = (\square, \varnothing) \rhd \mu_1 = (\varnothing, \square)$, and hence $\Lambda_{0,1}(1), \Lambda_{1,1}(1)$ are saturated, while $\Lambda_{1,0}(1)$ is not. The cardinality of $|\mathcal{T}_0(\mu_\bullet, \mu_\bullet)|$ is given as below:

$$|\mathcal{T}_0(\mu_1, \mu_1)| = 1 = |\mathcal{T}_0(\mu_2, \mu_1)| = |\mathcal{T}_0(\mu_2, \mu_2)|, \quad |\mathcal{T}_0(\mu_1, \mu_2)| = 0.$$

Note that $\mathcal{T}_0(\mu_1, \mu_2)$ is empty since the only μ_2-tableau of type μ_1 is $(\varnothing, \boxed{1}\boxed{2})$, which violates (S3). Hence, the dimensions of these cyclotomic Schur algebras are

$$S_q(\Lambda_{0,1}(1)) = 1, \quad S_q(\Lambda_{0,1}(1)) = 3.$$

For $d = 2$, the dominance order in $\Lambda_2^+(2)$ is given by

$$\lambda_5 = (\square\square, \varnothing) \rhd \lambda_4 = \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing\right) \rhd \lambda_3 = (\square, \square) \rhd \lambda_2 = (\varnothing, \square\square) \rhd \lambda_1 = \left(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right).$$

The sets $\Lambda_{0,2}(2), \Lambda_{1,2}(2)$, and $\Lambda_{2,2}(2)$ are saturated. The cardinality of $|\mathcal{T}_0(\lambda_\bullet, \lambda_\bullet)|$ is given in the following table

type\shape	λ_5	λ_4	λ_3	λ_2	λ_1
λ_5	1	0	0	0	0
λ_4	1	1	0	0	0
λ_3	1	1	1	0	0
λ_2	1	0	1	1	0
λ_1	1	1	2	1	1

Hence, the dimensions are

$$\dim S_q(\Lambda_{0,2}(2)) = 3, \quad \dim S_q(\Lambda_{1,2}(2)) = 7, \quad \dim S_q(\Lambda_{2,2}(2)) = 15.$$

Recall that $\dim S_q^{\mathrm{B}}(2,d) = d+1$ for all d, hence the algebras S_q^{B} and $S_q(\Lambda)$ small ranks do not match in an obvious way.

9.3. **Sakamoto-Shoji Algebras.** The cyclotomic Hecke algebra $\mathbb{H}(r,1,d)$ does admit a Schur-type duality (cf. [SS99]) with the algebra $U_q(\mathfrak{gl}_{n_1} \times \ldots \times \mathfrak{gl}_{n_r})$ where $n_1 + \ldots + n_r = n$. Hence, it specializes to the following double centralizer properties, for $a + b = n$:

$$\begin{array}{ccccc} U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b) & & & & \\ \downarrow & & & & \\ S_q^{\mathrm{B}}(a,b,d) & \curvearrowright & T(a,b,d) = (K^n)^{\otimes d} & \curvearrowleft & \mathcal{H}_q^{\mathrm{B}}(d) \end{array}$$

We will see in (9.3.4) that T_0 acts as a scalar multiple on $T(a,b,d)$, which is different from our T_0-action (2.3.2). Consquently, the duality is different from the geometric one. We could not locate an identification between $S_q^{\mathrm{B}}(a,b,d)$ and $S_q(\Lambda)$ for some Λ in the literature.

Now we set up the compatible version of the cyclotomic Schur duality introduced in [SS99]. Let $R' = \mathbb{Q}(Q, q', u_1, u_2)$, and let $\mathbb{H}_{d,2}$ be the the R'-algebra generated by $a_1, \ldots, a_d$ subject to the relations below, for $2 \leqslant i \leqslant d, 1 \leqslant j < k-1 \leqslant d-1$:

$$(a_1 - u_1)(a_1 - u_2) = 0, \quad (a_i - q')(a_i + (q')^{-1}) = 0, \tag{9.3.1}$$

$$(a_1 a_2)^2 = (a_2 a_1)^2, \quad a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \quad a_k a_j = a_j a_k. \tag{9.3.2}$$

With the identifications below one has the following result.

$$a_i \leftrightarrow T_{i-1}, \quad q' \leftrightarrow q^{-1} \tag{9.3.3}$$

Proposition 9.3.1. *The type B Hecke algebra $\mathcal{H}_{Q,q}^{\mathrm{B}}(d)$ is isomorphic to the algebra $\mathbb{H}_{d,2}$ at the specialization $u_1 = -Q, u_2 = Q^{-1}$.*

Let $T_Q(a,b,d) = V_{a,b}^{\otimes d}$ where $V_{a,b} = K^a \oplus K^b$ is the natural representation of $U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b)$ with bases $\{v_1^{(1)}, \ldots, v_a^{(1)}\}$ of K^a and $\{v_1^{(2)}, \ldots, v_b^{(2)}\}$ of K^b. The tensor space $T_Q(a,b,d)$ admits an obvious action of the type A Hecke algebra generated by $T_1, \ldots, T_{d-1}$. The T_0-action on $T(a,b,d)$ is more subtle as defined by

$$T_0 = T_1^{-1} \circ \ldots \circ T_{d-1}^{-1} \circ S_{d-1} \circ \ldots \circ S_1 \circ \varpi \in \mathrm{End}(T(a,b,d)), \tag{9.3.4}$$

where ϖ is given by

$$\varpi(x_1 \otimes \ldots \otimes x_d) = \begin{cases} -Q x_1 \otimes \ldots \otimes x_d & \text{if } x_1 = v_i^{(1)} \text{ for some } i; \\ Q^{-1} x_1 \otimes \ldots \otimes x_d & \text{if } x_1 = v_i^{(2)} \text{ for some } i, \end{cases} \tag{9.3.5}$$

and that S_i is given by

$$S_i(x_1 \otimes \ldots \otimes x_d) = \begin{cases} T_i(x_1 \otimes \ldots \otimes x_d) & \text{if } x_i, x_{i+1} \text{ both lies in } K^a \text{ or } K^b; \\ \ldots x_{i-1} \otimes x_{i+1} \otimes x_i \otimes x_{i+2} \otimes \ldots & \text{otherwise.} \end{cases} \tag{9.3.6}$$

Define

$$S_{Q,q}^{\mathrm{B}}(a,b,d) = \mathrm{End}_{\mathcal{H}_{Q,q}^{\mathrm{B}}(d)} \left(T_Q(a,b,d) \right). \tag{9.3.7}$$

It is proved in [SS99] that there is a Schur duality as below:

$$\begin{array}{ccccc} U_q(\mathfrak{gl}_a \times \mathfrak{gl}_b) & & & & \\ \downarrow & & & & \\ S^{\mathrm{B}}_q(a,b,d) & \curvearrowright & T(a,b,d) & \curvearrowleft & \mathcal{H}^{\mathrm{B}}_q(d) \end{array}$$

Example 9.3.2. Let $a = b = 1, d = 2$. Then $T_Q(1,1,2)$ has a basis $\{v := v_1^{(1)}, w := v_1^{(2)}\}$. The T_0-action is given by

$$\begin{aligned} (v \otimes v)T_0 &= -Qv \otimes v, \\ (v \otimes w)T_0 &= -Qv \otimes w, \\ (w \otimes v)T_0 &= Q^{-1}(w \otimes v + (q^{-1} - q)v \otimes w), \\ (w \otimes w)T_0 &= Q^{-1} w \otimes w. \end{aligned}$$

Note that this is essentially different from the T_0-action for the coideal Schur algebra given in (2.3.2).

9.4. **Slim cyclotomic Schur Algebras.** The slim cyclotomic Schur algebra $S_{(u_1,\ldots,u_r)}(n,d)$ introduced in [DDY18] is another attempt to establish a Schur duality for the cyclotomic Hecke algebra $\mathbb{H}(r,1,d)$. When $r = 2$, the algebra $S_{(u_1,u_2)}(n,d)$ has the same dimension as the coideal q-Schur algebra $S^{\mathrm{B}}_{Q,q}(2n,d)$; while there is no counterparts for the algebra $S^{\mathrm{B}}_{Q,q}(2n+1,d)$.

It is conjectured in [DDY18] that there is a weak Schur duality between the cyclotomic Hecke algebras and certain Hopf subalgebras $U_q(\widehat{\mathfrak{sl}}_n)^{(t)}$ of $U_q(\widehat{\mathfrak{gl}}_n)$ for an integer t to be determined. In our setting it can be phrased as follows:

$$\begin{array}{ccccccccc} U_q(\widehat{\mathfrak{gl}}_n) & \supsetneq & U_q(\widehat{\mathfrak{sl}}_n)^{(t)} & & & & & & \\ & & \downarrow & & & & & & \\ & & S^{\hat{\mathrm{A}}}_q(n,d) & \to & S_{(q,q)}(n,d) & \curvearrowright & \Omega^{\otimes d} & \curvearrowleft & \mathcal{H}^{\mathrm{B}}_q(d) \end{array}$$

Here $S_{(q,q)}(n,d) = \mathrm{End}_{\mathcal{H}^{\mathrm{B}}_q(d)}(T_{(q,q)}(n,d))$ is the centralizer algebra of the $\mathcal{H}^{\mathrm{B}}_q(d)$-action on a finite dimensional q-permutation module $T_{(q,q)}(n,d)$, while Ω is the (infinite-dimensional) natural representation of $U_q(\widehat{\mathfrak{gl}}_n)$.

We remark that it is called a weak duality in the sense that there are epimorphisms $U_q(\widehat{\mathfrak{sl}}_n)^{(t)} \twoheadrightarrow S_{(q,q)}(n,d)$ and $\mathcal{H}^{\mathrm{B}}_q(d) \twoheadrightarrow \mathrm{End}_{S_{(q,q)}(n,d)}(\Omega^{\otimes d})$; while it is not a genuine double centralizer property.

References

[AST18] H. H. Andersen, C. Stroppel and D. Tubbenhauer, *Cellular structures using U_q-tilting modules*, Pacific J. Math. 292 (2018), 21–59.

[B17] H. Bao, *Kazhdan-Lusztig theory of super type D and quantum symmetric pairs*, Represent. Theory 21 (2017), 247–276.

[BKLW18] H. Bao, J. Kujawa, Y. Li and W. Wang, *Geometric Schur duality of classical type*, Transform. Groups 23 (2018), 329–389.

[BLM90] A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*, Duke Math. J., 61 (1990), 655–677.

[BW13] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, Astérisque, 402, (2018).

[BWW18] H. Bao, W. Wang and H. Watanabe, *Multiparameter quantum Schur duality of type B*, Proc. Amer. Math. Soc., 146 (2018), no. 9, 3203 – 3216,

[CPS88] E. Cline, B. Parshall, and L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math., 391 (1988) 85 – 99.

[DDY18] B. Deng, J. Du and G. Yang, *Slim cyclotomic q-Schur algebras*, arXiv:1803.09185.

[DEMN99] S. R. Doty, K. Erdmann, S. Martin and D. K. Nakano, *Representation type for Schur algebras*, Math. Zeit., 232 (1999), 137–182.

[DEN04] S. R. Doty, K. Erdmann and D. K. Nakano, *Extensions of modules over Schur algebras, symmetric groups and Hecke algebras*, Algebr. Represent. Theory, 7 (2004), 67 –99.

[DN98] S. R. Doty and D. K. Nakano, *Semisimplicity of Schur algebras*, Math. Proc. Cambridge Phil. Soc., 124 (1998), 15–20.

[DJ89] R. Dipper and G. James, *The q-Schur algebra*, Proc. London Math. Soc., 59 (1989), 23 –50.

[DJ92] R. Dipper and G. James, *Representations of Hecke algebras of type B_n*, J. Algebra, 146 (1992), 454 – 481.

[DJM98] R. Dipper, G. D. James and A. Mathas, *The (Q, q)-Schur algebra*, Proc. Lond. Math. Soc., 77 (1998), 327–361.

[Do98] S. Doty, *Polynomial representations algebraic monoids and Schur algebras of classical type*, J. Pure Appl. Algebra, 123 (1998), 165–199.

[DS00] J. Du and L. Scott, *The q-Schur^2 algebra*, Trans. Amer. Math. Soc., 352 (2000), no. 9, 4325 – 4353.

[E93] K. Erdmann, *Schur algebras of finite type*, Quarterly J. Math. Oxford, 44 (1993), 17–41.

[EG02] P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space and deformed Harish-Chandra homomorphism*, Inv. Math., 147 (2002), 243–348.

[EN01] K. Erdmann and D. K. Nakano, *Representation type of q-Schur algebras*, Trans. Amer. Math. Soc., 353 (2001), no. 12, 4729 – 4756.

[ES18] M. Ehrig and C. Stroppel, *Nazarov–Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, Adv. Math., 331 (2018),

[FL15] Z. Fan and Y. Li, *Geometric Schur duality of classical type, II*, Trans. Amer. Math. Soc., Series B. 2 (2015), 51–92.

[FNP01] V. Futorny, D.K. Nakano and R.D. Pollack, *Representation type of the blocks of category $\mathcal{O}$*, Quart. J. Math. Oxford, 52 (2001), 285–305.

[FRT90] L. Faddeev, N. Reshetikhin and L. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J., 1 (1990), 193–225.

[Ge07] M. Geck, *Hecke algebras of finite type are cellular*, Invent. Math., 169 (2007), 501–517.

[GGOR03] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math., 154 (2003), 617–651.

[GL96] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math., 123 (1996), 1–34.

[Gr80] J. A. Green, *Polynomial representations of GL_n*, Lecture Notes in Math., 830, Springer, Berlin, 1980.

[Gr97] R. Green, *Hyperoctahedral Schur algebras*, J. Algebra, 192 (1997), 418–438.

[HN04] D. J. Hemmer and D. K. Nakano, *Specht filtrations for Hecke algebras of type A*, J. London Math. Soc., 69 (2004), 623–638.

[H11] J. Hu, *BMW algebra, quantized coordinate algebra and type C Schur-Weyl duality*, Represent. Theory, 15 (2011), 1–62.

[Jim86] M. Jimbo, *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys., 11 (1986), 247–252.

[KX98] S. König and C. Xi, *On the structure of cellular algebras*, Canad. Math. Soc. Conf. Proc., 24 (1998), 365–386.

[Le17] G. Letzter, *Cartan Subalgebras for Quantum Symmetric Pair Coideals*, Representation Theory (to appear), arXiv:1705.05958.

[LL18] C. Lai and L. Luo, *Schur algebras and quantum symmetric pairs with unequal parameters*, arXiv:1808.00938.

[PW91] B. Parshall, and J. Wang, *Quantum linear groups*, Mem. Amer. Math. Soc., 439 (1991).

[Ro08] R. Rouquier *q-Schur algebras and complex reflection groups*, Mosc. Math. J., 8 (2008) 119–158.

[SS99] M. Sakamoto, and T. Shoji, *Schur-Weyl reciprocity for Ariki-Koike algebras*, J. Algebra, 221 (1998), 293–314.

Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: cjlai@uga.edu

Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: nakano@math.uga.edu

Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: ziqing@uga.edu