Nonexistence of nontrivial tight 8-designs

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Abstract

Tight t-designs are t-designs whose sizes achieve the Fisher type lower bound. We give a new necessary condition for the existence of nontrivial tight designs and then use it to show that there do not exist nontrivial tight 8-designs.
Keywords: Tight design, Intersection number, product of consecutive integers.

1 Introduction

Let t, v, k and λ be positive integers, and $[v] := \{1, 2, \ldots, v\}$. A t-(v, k, λ) *design*, or simply a t-*design*, is a set $\mathcal{B} \subseteq \binom{[v]}{k}$ satisfying for all $T \in \binom{[v]}{t}$,

$$\#\{b \in \mathcal{B} : \; T \subseteq b\} = \lambda.$$

A design is *trivial* if it consists of all k-element subsets of $[v]$. It is easy to show that when $v \leq k + t$, a t-(v, k, λ) design is trivial. So, nontrivial t-(v, k, λ) designs have the property that $v > k + t$.

In 1975, D. Ray-Chaudhuri and R. Wilson [12] proved a *Fisher type lower bound* on the size of a nontrivial design: For a nontrivial t-design $\mathcal{B}$,

$$|\,\mathcal{B}\,| \geq \begin{cases} \binom{v}{e}, & \text{if } t = 2e, \\ 2\binom{v-1}{e-1}, & \text{if } t = 2e - 1, \end{cases}$$

where $e := \lceil t/2 \rceil$. A nontrivial t-design is *tight* provided that the size of the design achieves the lower bound above. Since $\lambda\binom{v}{t} = \binom{k}{t}|\,\mathcal{B}\,|$ for a t-(v, k, λ) design $\mathcal{B}$, if $\mathcal{B}$ is tight, then λ is determined by t, v and k. Given a nontrivial tight t-design, it is of *strength* t, namely, it is a t-design but not a $(t + 1)$-design. The *complementary design* of a nontrivial t-(v, k, λ) design $\mathcal{B}$ consists of blocks $\{[v] \setminus b : \; b \in \mathcal{B}\}$. It is a t-$(v, v - k, \lambda')$ design for some positive integer λ'. The complementary design of a nontrivial tight t-design is also a nontrivial tight t-design.

It is easy to show that there exists a unique tight 1-(v, k, λ) design up to isomorphism for each k, the 1-$(2k, k, 1)$ design. Tight 2-designs are also called *symmetric designs*. The case $t = 2$ is quite different from the other cases. Finite projective planes and Hadamard matrices give two infinite families of nontrivial tight 2-designs [7]. A complete classification of tight 2-designs is not yet known.

In 1975, D. Ray-Chaudhuri and R. Wilson [12] proved the nonexistence of nontrivial tight t-designs when $t \geq 3$ is odd. In 1977, E. Bannai [1] succeeded in proving that for any $t = 2e$ with $e \geq 5$, there exist only finitely many nontrivial tight t-designs. In an 1977 unpublished paper [2], E. Bannai and T. Ito proved the finiteness of the number of nontrivial tight 8-designs. In 1977, the nonexistence of nontrivial tight 6-designs was proved by C. Peterson in [11]. In 1979, A. Bremner showed in [3] that there are only two nontrivial tight 4-designs up to isomorphism, which have parameters 4-$(23, 7, 1)$ and 4-$(23, 16, 52)$. A second proof of this fact was given by R. Stroeker [14]. Note that these two nontrivial tight 4-designs are complementary to each other. In 2013, P. Dukes and J. Short-Gershman proved the nonexistence of nontrivial tight t-designs for $t \in \{10, 12, \ldots, 18\}$ in [5].

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In this article, we first give a new necessary condition for the existence of nontrivial tight designs and then use it to prove the nonexistence of nontrivial tight 8-designs. To state our new necessary condition, let us recall the following notation. For nonnegative integer n, the *n-th rising factorial* and the *n-th falling factorial* of x are

$$x^{\overline{n}} := \prod_{i=0}^{n-1}(x+i) \quad \text{and} \quad x^{\underline{n}} := \prod_{i=0}^{n-1}(x-i),$$

respectively. We adopt the convention that $x^{\overline{0}} = x^{\underline{0}} = 1$.

Theorem 1.1. *Let e be a positive integer. For each $0 \leq i \leq e$, let*

$$h_i := \frac{(k-e)^{\overline{i+1}}}{(v-2e+1)^{\overline{i}}} \in \mathbb{Q}(v,k). \tag{1.1}$$

There exists a nonzero integer c_e such that if there exists a nontrivial tight $2e$-(v_0,k_0,λ) design, then for every $0 \leq i \leq e$, the rational function $c_e h_i$ takes an integer value at the point (v_0,k_0). Moreover, c_e can be chosen so that it has only prime factors no greater than $2e-2$.

An explicit c_e is given in Lemma 4.1 as $c_{e,0}$, but it is far from being optimal. In practice, a smaller value c_e can be computed explicitly for small e. Proposition 4.3 shows that we can choose $c_4 = 288$ in Theorem 1.1, which leads us to the nonexistence of nontrivial tight 8-designs.

Theorem 1.2. *There do not exist nontrivial tight 8-designs.*

The proofs of Theoremss 1.1 and 1.2 can be found in Sections 4 and 6, respectively.

2 Intersection numbers

Given a design $\mathcal{B}$, the integers in the set

$$\left\{ |b \cap b'| : \ \{b,b'\} \in \binom{\mathcal{B}}{2} \right\}$$

are called the *intersection numbers* of the design. The following proposition shows that we can determine all intersection numbers of a nontrivial tight $2e$-(v_0,k_0,λ) design just using v_0, k_0 and e.

Proposition 2.1 ([4, implicit], [12, p. 743], [11, implicit], [1, Proposition 1], [5, Proposition 1.1])**.** *For a nontrivial tight $2e$-(v_0,k_0,λ) design, the zeros of the polynomial $\Phi_{e,v_0,k_0} \in \mathbb{Q}[x]$ of degree e are distinct integers and the zeros coincide with the intersection numbers of the design, where*

$$\Phi_{e,v_0,k_0}(x) := \sum_{i=0}^{e}(-1)^{e-i}\frac{\binom{v_0-e}{i}\binom{k_0-i}{e-i}\binom{k_0-i-1}{e-i}}{\binom{e}{i}}\binom{x}{i}.$$

For each $0 \leq i \leq e$, let

$$p_i := \binom{e}{i}\frac{(k-e)^{\overline{i}}(k-e+1)^{\overline{i}}}{(v-2e+1)^{\overline{i}}} \in \mathbb{Q}(v,k). \tag{2.1}$$

Corollary 2.2. *For a tight $2e$-(v_0,k_0,λ) design and each $0 \leq i \leq e$, the value of the rational function p_i at the point (v_0,k_0) is an integer.*

Proof. Let

$$\Psi_{e,v_0,k_0} := \frac{e!}{\binom{v_0-e}{e}}\Phi_{e,v_0,k_0},$$

which is a monic polynomial in $\mathbb{Q}[x]$. By Proposition 2.1, all roots of Ψ_{e,v_0,k_0}, which is a constant multiple of Φ_{e,v_0,k_0}, are intersection numbers, which are all integers. Thus, the monic polynomial Ψ_{e,v_0,k_0} has integral coefficients. We proceed with the following calculation:

$$\begin{aligned}\Psi_{e,v_0,k_0}(x) &= \sum_{i=0}^{e}(-1)^{e-i}\binom{e}{e-i}\frac{(k_0-e)^{\overline{e-i}}(k_0-e+1)^{\overline{e-i}}}{(v_0-2e+1)^{\overline{e-i}}}\cdot x^{\underline{i}}\\ &= \sum_{i=0}^{e}(-1)^{i}p_i(v_0,k_0)\cdot x^{\underline{e-i}}.\end{aligned}$$

For each i, the polynomial $x^{\underline{e-i}} \in \mathbb{Z}[x]$ is a monic polynomial of degree $e-i$. Therefore, all $p_i(v_0,k_0)$ are integers. $\square$

3 Some ideals

In this section, we encounter some ideals I and try to find an explicit positive integer n in the intersection of I and $\mathbb{Z}$. In practice, for an explicitly given ideal $I \subseteq \mathbb{Z}[x_1,\dots,x_m]$, we could compute the Gröbner basis of I over $\mathbb{Z}$ to get the intersection $I\cap\mathbb{Z}$. More precisely, if the Gröbner basis contains an integer, then that integer is a generator of $I\cap\mathbb{Z}$, and if it does not contain an integer, then $I\cap\mathbb{Z}$ is trivial. Our key result in this section is Lemma 3.3, which is used in the proof of Lemma 4.1, and this latter lemma is an essential ingredient in the proof of Theorem 1.1. The smaller n is, the smaller the positive integer c_e in Theorem 1.1 is.

Lemma 3.1. *Let d be a nonnegative integer and x_0, x_1 be integers. Let $f_d := (x+x_0)^{\overline{d}}$ and $g_d := (x+x_1)^{\underline{d}}$ be polynomials in $\mathbb{Z}[x]$, and consider the ideal $I_{d,x_0,x_1} := \langle f_d, g_d\rangle$ in $\mathbb{Z}[x]$. The following statements hold.*

(i) *The intersection $I_{d,x_0,x_1}\cap\mathbb{Z}$ is trivial if and only if $x_1-x_0\in[0,2d-2]$, where $[0,2d-2]:=\emptyset$ when $d=0$.*

(ii) *$(x_1-x_0)^{\underline{2d-1}} \in I_{d,x_0,x_1}\cap\mathbb{Z}$.*

Proof. Let $I := I_{d,x_0,x_1}$.
(i) It suffices to prove that $\mathbb{Q}I\cap\mathbb{Q}$ is trivial if and only if $x_1-x_0\in[0,2d-2]$. The ring $\mathbb{Q}[x]$ is a principal ideal domain, so the greatest common divisor of f_d and g_d is a generator of $\mathbb{Q}I$. The zeros of f_d are $[-x_0-d+1,x_0]\cap\mathbb{Z}$, and the zeros of g_d are $[-x_1,-x_1+d-1]\cap\mathbb{Z}$. So, f_d and g_d have a common factor of positive degree if and only if $[-x_0-d+1,x_0]\cap[-x_1,-x_1+d-1]\neq\emptyset$, which is equivalent to $x_1-x_0\in[0,2d-2]$. Thus, $\mathbb{Q}I\cap\mathbb{Q}$ is empty if and only if $x_1-x_0\in[0,2d-2]$.
(ii) Let $z := x_1-x_0$, so that $f_d=(x+x_0)^{\overline{d}}$ and $g_d=(x+z+x_0)^{\underline{d}}$. Now, regard f_d and g_d as polynomials in $\mathbb{Z}[z][x]$, which is the ring of polynomials in x with coefficients in $\mathbb{Z}[z]$, and regard I as an ideal in $\mathbb{Z}[z][x]$. It suffices to prove that $z^{\underline{2d-1}}\in I\cap\mathbb{Z}[z]$.

Let $f_d' := x^{\overline{d}}\in\mathbb{Z}[x]$, $g_d' := z^{\underline{d}}\in\mathbb{Z}[z]$ and $J := \langle f_d', g_d'\rangle$ be an ideal in $\mathbb{Z}[z][x]$. Then, a linear change of variable gives a ring isomorphism $\mathbb{Z}[z][x]/I\cong\mathbb{Z}[z][x]/J$. Evaluation at (α,β), where α runs over zeros of $g_d'\in\mathbb{Z}[z]$ and β runs over zeros of $f_d'\in\mathbb{Z}[x]$, gives a ring monomorphism $\mathbb{Z}[z][x]/J\hookrightarrow\mathbb{Z}^{d^2}$. Now we have a chain of ring monomorphisms

$$\mathbb{Z}[z]/(I\cap\mathbb{Z}[z])\hookrightarrow\mathbb{Z}[z][x]/I\cong\mathbb{Z}[z][x]/J\hookrightarrow\mathbb{Z}^{d^2},$$

where the first monomorphism is induced from the inclusion $\mathbb{Z}[z]\hookrightarrow\mathbb{Z}[z][x]$. Since $\mathbb{Z}^{d^2}$ is a reduced ring, $\mathbb{Z}[z]/(I\cap\mathbb{Z}[z])$ is reduced as well, which implies that $I\cap\mathbb{Z}[z]$ is a radical ideal of $\mathbb{Z}[z]$.

The resultant of f_d and g_d, regarded as polynomials in x with coefficients in $\mathbb{Z}[z]$, is

$$\operatorname{res}_x(f_d,g_d)=\prod_{i=0}^{2d-1}(z-i)^{\min\{i+1,2d-i\}}\in\mathbb{Z}[z].$$

Since $\operatorname{res}_x(f_d,g_d)\in I\cap\mathbb{Z}[z]$ and $I\cap\mathbb{Z}[z]$ is a radical ideal, the square-free part of the resultant, $z^{\underline{2d-1}}$, is in $I\cap\mathbb{Z}[z]$. $\square$

Let u_{d,x_0,x_1} be the nonnegative generator of $I_{d,x_0,x_1} \cap \mathbb{Z}$, and let

$$v_{d,x_0,x_1} := \begin{cases} \frac{(x_1-x_0)^{\underline{2d-1}}}{u_{d,x_0,x_1}}, & u_{d,x_0,x_1} \neq 0, \\ 0, & u_{d,x_0,x_1} = 0. \end{cases}$$

According to Lemma 3.1, u_{d,x_0,x_1} divides $(x_1-x_0)^{\underline{2d-1}}$, hence v_{d,x_0,x_1} is always an integer. It seems that the generator u_{d,x_0,x_1} is very close to $(x_1-x_0)^{\underline{2d-1}}$, in the sense that v_{d,x_0,x_1} is much smaller than u_{d,x_0,x_1} and $(x_1-x_0)^{\underline{2d-1}}$. Based on computations, we propose Conjecture 3.2, and Table 3.1 gives some evidence for $d = 3$ or 4.

$x_1 - x_0$	u_{4,x_0,x_1}	$(x_1-x_0)^{\underline{7}}$	v_{4,x_0,x_1}	u_{3,x_0,x_1}	$(x_1-x_0)^{\underline{5}}$	v_{3,x_0,x_1}
-1	2520	-5040	-2	120	-120	-1
-2	10080	-40320	-4	120	-720	-6
-3	90720	-181440	-2	840	-2520	-3
-4	30240	-604800	-20	3360	-6720	-2
-5	166320	-1663200	-10	5040	-15120	-3
-6	997920	-3991680	-4	5040	-30240	-6
-7	4324320	-8648640	-2	55440	-55440	-1
-8	4324320	-17297280	-4	15840	-95040	-6
-9	3243240	-32432400	-10	51480	-154440	-3
-10	2882880	-57657600	-20	120120	-240240	-2

Table 3.1: Comparison of u_{d,x_0,x_1} and $(x_1-x_0)^{\underline{2d-1}}$ for $d = 3, 4$. The columns for v_{4,x_0,x_1} and v_{3,x_0,x_1} are periodic with periods 10 and 6, respectively.

Conjecture 3.2. Let d be a nonnegative number. The following statements hold:

(i) $v_{d,1,0}$ divides v_{d,x_0,x_1}.

(ii) $v_{d,1,0}$ divides the $(d-2)$-th generalized Catalan number with respect to function $f(n) := n/\mathrm{rad}(n)$ in the sense of [6].

(iii) The sequence $\{v_{d,x_0,0}\}_{x_0 \geq 1}$ is periodic in the sense that for all $x_0 \geq 0$, we have $v_{d,x_0+p,0} = v_{d,x_0,0}$ for some positive integer p depending on d. Moreover, the period is $1, 2, 6, 10, 70, 126, 154, 286$ when d is $1, 2, 3, 4, 5, 6, 7, 8$, respectively.

The following lemma studies the two variable version of Lemma 3.1.

Lemma 3.3. *Let d be a nonnegative integer and x_0, y_0, x_1, y_1 be integers. For each $0 \leq i \leq d$, let $f_{d,i} := (x+x_0)^{\overline{i}}(y+y_0)^{\overline{d-i}}$ and $g_{d,i} := (x+x_1)^{\underline{i}}(y+y_1)^{\underline{d-i}}$ be polynomials in $\mathbb{Z}[x,y]$. Consider the ideal*

$$I_{d,x_0,y_0,x_1,y_1} := \langle f_{d,i}, g_{d,i} : \ 0 \leq i \leq d \rangle$$

in $\mathbb{Z}[x,y]$. Then, the ideal $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z}$ contains the three ideals $I_{d,x_0,x_1} \cap \mathbb{Z}$, $I_{d,y_0,y_1} \cap \mathbb{Z}$ and $I_{d,x_0+y_0,x_1+y_1} \cap \mathbb{Z}$, which are defined as in Lemma 3.1.

Proof. Let $J := I_{d,x_0,y_0,x_1,y_1}$. The result holds trivially when $d = 0$. Assume that $d \geq 1$. Since $f_{d,d} = (x+x_0)^{\overline{d}}$ and $g_{d,d} = (x+x_1)^{\underline{d}}$, we have $J \cap \mathbb{Z} \supseteq \langle f_{d,d}, g_{d,d} \rangle \cap \mathbb{Z} = I_{d,x_0,x_1} \cap \mathbb{Z}$. Similarly, since $f_{d,0} = (y+y_0)^{\overline{d}}$ and $g_{d,0} = (y+y_1)^{\underline{d}}$, we have $J \cap \mathbb{Z} \supseteq \langle f_{d,0}, g_{d,0} \rangle \cap \mathbb{Z} = I_{d,y_0,y_1} \cap \mathbb{Z}$.

The sum of $f_{d,i}$ and $f_{d,i+1}$ is in the ideal J:

$$\begin{aligned} J \ni & f_{d,i} + f_{d,i+1} \\ =& (x+x_0)^{\overline{i}}(y+y_0)^{\overline{d-i-1}}((x+x_0+i)+(y+y_0+d-i-1)) \\ =& (x+x_0)^{\overline{i}}(y+y_0)^{\overline{d-i-1}}(x+y+x_0+y_0+d-1) \\ =& (x+y+x_0+y_0+d-1)f_{d-1,i}. \end{aligned}$$

Using induction on the equation above, we obtain

$$
\begin{aligned}
(x+y+x_0+y_0)^{\overline{d}} =& (x+y+x_0+y_0+d-1)^{\underline{d}} f_{0,0} \\
=& \sum_{i=0}^{d} \binom{d}{i} f_{d,i} \in J.
\end{aligned} \tag{3.1}
$$

We do the same thing for g's and get

$$
\begin{aligned}
J \ni & g_{d,i} + g_{d,i+1} \\
=& (x+x_1)^{\underline{i}} (y+y_1)^{\underline{d-i-1}} ((x+x_1-i)+(y+y_1-d+i+1)) \\
=& (x+x_1)^{\underline{i}} (y+y_1)^{\underline{d-i-1}} (x+y+x_1+y_1-d+1) \\
=& (x+y+x_1+y_1-d+1) g_{d,i}.
\end{aligned}
$$

Again using induction, we have

$$
\begin{aligned}
(x+y+x_1+y_1)^{\underline{d}} =& (x+y+x_1+y_1-d+1)^{\overline{d}} g_{0,0} \\
=& \sum_{i=0}^{d} \binom{d}{i} g_{d,i} \in J.
\end{aligned} \tag{3.2}
$$

Therefore, combining Eqs. (3.1) and (3.2), it follows that $J \cap \mathbb{Z} \supseteq \langle (x+y+x_0+y_0)^{\overline{d}}, (x+y+x_1+y_1)^{\underline{d}} \rangle \cap \mathbb{Z} = I_{d,x_0+y_0,x_1+y_1} \cap \mathbb{Z}$. $\square$

$y_1 - y_0$																	
7	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	2520	
6		2520	1080	360	720	1080	360	360	0	360	360	1080	720	360	1080	2520	
5			2520	360	240	720	360	120	0	0	120	360	720	240	360	2520	
4				2520	720	720	144	72	0	0	0	72	144	720	720	2520	
3					2520	1080	360	72	0	0	0	0	72	360	1080	2520	
2						2520	360	120	0	0	0	0	0	120	360	2520	
1							2520	360	0	0	0	0	0	0	360	2520	
0								2520	0	0	0	0	0	0	0	2520	
-1									2520	360	120	72	72	120	360	2520	
-2										2520	360	360	144	360	360	2520	
-3											2520	1080	720	720	1080	2520	
-4												2520	720	240	720	2520	
-5													2520	360	360	2520	
-6														2520	1080	2520	
-7															2520	2520	
-8																2520	
	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	$x_1 - x_0$

Table 3.2: The generators of the intersections of I_{4,x_0,y_0,x_1,y_1} and $\mathbb{Z}$.

We list the intersection of the ideal I_4 and $\mathbb{Z}$ in Table 3.2. We can read some patterns from the table. For instance, we see some constant lines, and the table is symmetric with respect to the three medians of the triangle made by zeros. Computations support the following conjecture.

Conjecture 3.4. Let I_{d,x_0,y_0,x_1,y_1} be the ideal defined in Lemma 3.3, and let u_{d,x_0,y_0,x_1,y_1} be the nonnegative generator of $I_{d,x_0,y_0,x_1,y_1} \cap \mathbb{Z}$. The following statements hold.

(i) $u_{d,0,0,x_1,y_1}$ is a multiple of $d!$.

(ii) $u_{d,0,0,x_1,y_1} = u_{d,0,0,y_1,x_1} = u_{d,0,0,x_1,2d-2-x_1-y_1} = u_{d,0,0,2d-2-x_1-y_1,y_1}$.

(iii) If $x_1 = 2d-1$, or $y_1 = 2d-1$, or $x_1 + y_1 = -1$, then $u_{d,0,0,x_1,y_1} = u_{d,1,0}$.

Motivated by the one-variable case in Lemma 3.1 and the two-variable case in Lemma 3.3, we conjecture below the existence of a similar phenomenon for an arbitrary number of variables. For an n-part partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of a nonnegative integer d and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, let

$$
\mathbf{x}^{\overline{\lambda}} := \prod_{i=1}^{n} x_i^{\overline{\lambda_i}} \quad \text{and} \quad \mathbf{x}^{\underline{\lambda}} := \prod_{i=1}^{n} x_i^{\underline{\lambda_i}}
$$

be the λ-*rising factorial* of $\mathbf{x}$ and the λ-*falling factorial* of $\mathbf{x}$, respectively.

Conjecture 3.5. Let d be a nonnegative integer and $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{Z}^n$. For each n-part partition λ of d, let $f_\lambda := (\mathbf{x}+\mathbf{x}_0)^{\overline{\lambda}}$ and $g_\lambda := (\mathbf{x}+\mathbf{x}_1)^{\underline{\lambda}}$ be polynomials in $\mathbb{Q}[\mathbf{x}]$. The ideal of $\mathbb{Q}[x]$ generated by f_λ and g_λ, where λ runs over all n-part partitions of d, is not the whole ring if and only if $\mathbf{x}_1 - \mathbf{x}_0 \geq \mathbf{0}$ and $\|\mathbf{x}_1 - \mathbf{x}_0\|_1 \leq 2d-2$.

4 Proof of Theorem 1.1

For each $e+1 \leq i \leq 2e$, let

$$\lambda_i := \frac{1}{e!} \frac{k^{\underline{i}}}{(v-e)^{\underline{i-e}}} \in \mathbb{Q}(v,k). \tag{4.1}$$

It is well known that a t-design $\mathcal{B}$ is also an i-design for each $i \leq t$. In particular, in the case where $\mathcal{B}$ is a tight $2e$-(v_0, k_0, λ) design, it is also an i-$(v_0, k_0, \lambda_i(v_0, k_0))$ design for each $e+1 \leq i \leq 2e$.

Lemma 4.1. *For every $0 \leq i \leq e$, the rational function $c_{e,i} h_i$ (see Eq. (1.1)), where*

$$c_{e,i} := (2e-2)! \prod_{j=i}^{e} j!,$$

is a $\mathbb{Z}[v,k]$-linear combination of $p_1, \dots, p_e$ (see Eq. (2.1)) and $\lambda_{e+1}, \dots, \lambda_{2e}$.

Proof. First, $h_0 = k - e \in \mathbb{Z}[v,k]$, so we only need to consider $h_1, \dots, h_e$. For each $1 \leq i \leq e$, let

$$f_i := (-1)^{i-1} e! \lambda_{e+i} = (-k+e+1)^{\overline{i-1}} (v-2e+1)^{\overline{e-i}} \cdot h_e,$$

and

$$g_i := (-1)^{i-1} \binom{e}{i}^{-1} (k)^{\underline{e-i}} p_i = (-k+e-1)^{\underline{i-1}} (v-e)^{\underline{e-i}} \cdot h_e.$$

By Lemma 3.3 applied with the variables $x = -k$, $y = v$ and the parameters $x_0 = e+1$, $y_0 = -2e+1$, $x_1 = e-1$, $y_1 = -e$ and $d = e-1$, the ideal in $\mathbb{Z}[v,k]$,

$$\begin{aligned} &\langle f_i/h_e, g_i/h_e : \ 1 \leq i \leq e \rangle \\ =&\langle (-k+e+1)^{\overline{i-1}} (v-2e+1)^{\overline{e-i}}, (-k+e-1)^{\underline{i-1}} (v-e)^{\underline{e-i}} : \ 1 \leq i \leq e \rangle \end{aligned}$$

contains the ideal $I_{d,x_0,x_1} \cap \mathbb{Z} = I_{e-1,e+1,e-1} \cap \mathbb{Z}$, which is defined in Lemma 3.1.

By Lemma 3.1, $(2e-2)! \in I_{e-1,e+1,e-1} \cap \mathbb{Z}$, therefore, $(2e-2)! h_e$ is a $\mathbb{Z}[v,k]$-linear combination of $f_1, \dots, f_e$ and $g_1, \dots, g_e$, hence $(2e-2)! e! h_e = c_{e,e} h_e$ is a $\mathbb{Z}[v,k]$-linear combination of $p_1, \dots, p_e$ and $\lambda_{e+1}, \dots, \lambda_{2e}$.

Let i be an arbitrary integer in $[1, e-1]$. In the quotient ring $\mathbb{Z}[k]/(k-e+i+1)$, we have $(k-e+1)^{\overline{i-1}} = (-i)^{\overline{i-1}} = (-1)^{i-1} i!$. So, we can express $(k-e+1)^{\overline{i-1}}$ as $(k-e+i+1)u_i + (-1)^{i-1} i!$ for some $u_i \in \mathbb{Z}[k]$. Then,

$$\begin{aligned} &(v-2e+i+1)u_i \cdot h_{i+1} \\ =&(k-e+i+1)u_i \cdot h_i \\ =&((k-e+1)^{\overline{i-1}} + (-1)^i i!) \cdot h_i \\ =&(k-e+1)^{\overline{i-1}} \cdot h_i + (-1)^i i! \cdot h_i \\ =&\binom{e}{i}^{-1} \cdot p_i + (-1)^i i! \cdot h_i, \end{aligned}$$

which shows that $c_{e,i} h_i = c_{e,i+1} i! h_i$ is a $\mathbb{Z}[v,k]$-linear combination of $c_{e,i+1} h_{i+1}$ and p_i. The result follows by an induction. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Assume that there exists a tight $2e$-(v_0, k_0, λ) design $\mathcal{B}$. By Corollary 2.2, p_i takes an integer value at (v_0, k_0) for every $1 \le i \le e$. The $2e$-design $\mathcal{B}$ is also an i-(v_0, k_0, λ_i) design for each $e+1 \le i \le 2e$, so λ_i in Eq. (4.1) takes an integer value at (v_0, k_0) as well. Due to Lemma 4.1, for every $0 \le i \le e$, the rational function $c_{e,i}h_i$ is a $\mathbb{Z}[v,k]$-linear combination of $p_1, \dots, p_e$ and $\lambda_{e+1}, \dots, \lambda_{2e}$, hence it takes an integer value at (v_0, k_0). The result follows from the fact that the integer $c_{e,i}$ is always a divisor of $c_{e,0}$, and $c_{e,0}$ has only prime factors no greater than $2e-2$. $\square$

Remark 4.2. When $e = 4$, we can give explicit $\mathbb{Z}[v,k]$-linear combinations for some multiples of h_i:

$$\begin{aligned}
q_4 := 24h_4 &= 288(v-7)(k-5)\lambda_6 - 96((v-15)(k-13)-64)\lambda_7 \\
&\quad -24(38v+3k-262)\lambda_8 + 6(k-5)k(k-1)(k-2)p_1 \\
&\quad -2((v-6)(k-5)+1)k(k-1)p_2 \\
&\quad +((v-9)(k-4)+2)kp_3 + (2v+3k-22)p_4, \\
q_3 := 144h_3 &= 6p_3 - (v-4)(k-5)q_4, \\
q_2 := 288h_2 &= -24p_2 + (v-5)q_3, \\
q_1 := 4h_1 &= p_1, \\
q_0 := h_0 &= k-4.
\end{aligned}$$

The expression for q_4 is obtained by calculating the Gröbner basis for a certain ideal, and the expressions for q_3 and q_2 follow from the proof of Lemma 4.1.

Using the formulas in Remark 4.2, Proposition 4.3 strengthens Theorem 1.1 in the case $e = 4$. The proof of Proposition 4.3 is very similar to the proof of Theorem 1.1 and is not presented here.

Proposition 4.3. *If there exists a nontrivial tight* 8-(v_0, k_0, λ) *design, then the rational functions* $q_0, \dots, q_4$ *take integer values at* (v_0, k_0)*. In particular, we could take* $c_4 = 288$ *in Theorem 1.1.*

Remark 4.4. If there exist infinitely many nontrivial tight $2e$-designs for a fixed e, then by Theorem 1.1, the h_i take bounded denominator value infinitely many times in the region $v > k + 2e$. Up to a linear change of variables, the expression h_i is equal to the expression appearing in Conjecture 4.5. It is immediate to see that if Conjecture 4.5 below holds for $n = e_0$, then for all $e \ge e_0$, there are only finitely many nontrivial tight $2e$-designs. Furthermore, any effective solution to Conjecture 4.5 could lead to effective bounds for v and k.

Conjecture 4.5. Let n be an integer at least 3. For every nonzero integer c, there are only finitely many pairs (x, y) of positive integers satisfying $y \ge x + 2$ and such that for every $1 \le i \le n$,

$$\frac{x^{\overline{i+1}}}{y^{\overline{i}}} \in \frac{1}{c}\mathbb{Z}\,.$$

(In other words, $\frac{x(x+1)}{y}$, $\frac{x(x+1)(x+2)}{y(y+1)}$, ..., have denominators that divide c.)

In the case where $n = 3$ and $c = 1$, there do not exist any such pairs for x up to 5 billion. More evidence is given in [10].

5 Asymptotic behavior of f_4

Let $f_4 \in \mathbb{Z}[v,k]$ be the polynomial of degree 13 given in §A. The polynomial f_4 was first found by E. Bannai and T. Ito in an unpublished work [2], and it is shown in [2] that for all but finitely many nontrivial tight 8-designs $\mathcal{B}$, $f_4(v_0, k_0) = 0$ where (v_0, k_0) are the parameters of $\mathcal{B}$. The polynomial was rediscovered by P. Dukes and J. Short-Gershman in [5] and they strengthen the result in [2] as follows.

Proposition 5.1 ([5, §4])**.** *If there exists a nontrivial tight* 8-(v_0, k_0, λ) *design, then* $f_4(v_0, k_0) = 0$.

The polynomial f_4 satisfies Runge's condition [13], so by Runge's theorem [13], it has finitely many integral solutions (an algebraic description of Runge's theorem could be found in [16, Chapter 2]). Using this approach, E. Bannai and T. Ito showed in [2] that there are only finitely many tight 8-designs.

Quantitative versions of Runge's theorem have been established, and using the results in [8] and [15], we can obtain the bounds $e^{e^{8600}}$ and $e^{e^{22}}$, respectively, for the size $\max(|v_0|,|k_0|)$ of an integral solution (v_0,k_0). The bounds are too large for any computer search to terminate.

The curve defined by the polynomial f_4 has an involution $(v,k)\mapsto(v,v-k)$, which corresponds to the construction of complementary designs. The geometric genus of the curve is 20, so by Faltings' theorem [9, Theorem E.0.1], the curve has only finitely many rational points. However, Faltings' theorem is not effective.

There are 32 known rational zeros of $f_4(v,k)$. They are $(-1,-3)$, $(-1,-2)$, $(-1,1)$, $(-1,2)$, $(11/5,1)$, $(11/5,6/5)$, $(4,0)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(13/3,2)$, $(13/3,7/3)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(27/5,2)$, $(27/5,13/5)$, $(27/5,14/5)$, $(27/5,17/5)$, $(6,2)$, $(6,11/4)$, $(6,3)$, $(6,13/4)$, $(6,4)$, $(125/19,54/19)$, $(125/19,71/19)$, $(7,3)$, $(7,4)$, $(15,2)$, $(15,13)$. However, we are unable to show that they are the only rational zeros. None of the known zeros could be realized by nontrivial tight 8-designs, since for nontrivial tight 8-(v_0,k_0,λ) designs, we have $v_0>k_0+8\geq 16$.

The main result of this section is Proposition 5.3, which describes the zeros of f_4 in the region $k\geq 10^5$ and $2k\leq v\leq \frac{4}{5}k^2$. In the remaining part of this section, the parameters v and k are assumed to be real numbers.

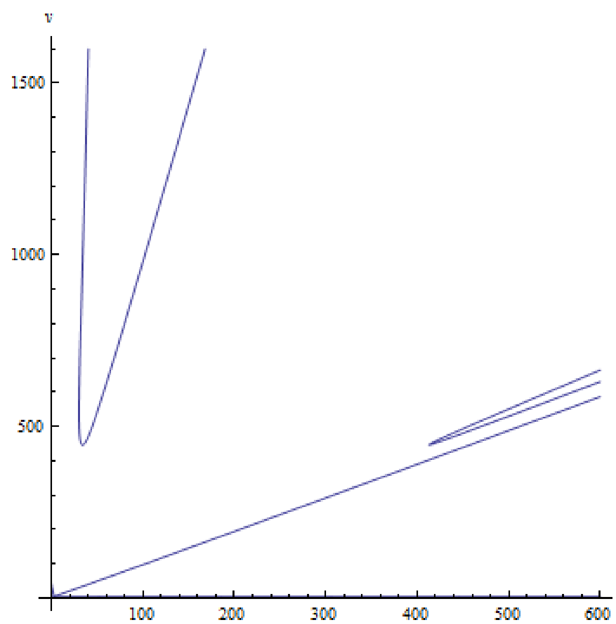

Figure 5.1: Real zeros of f_4 in the first quadrant.

We plot the reals zeros of f_4 in Figure 5.1. In the first quadrant, the plot shows six branches likely going to infinity (one branch is so close to the x-axis that it is indistinguishable from it in Figure 5.1). Some necessary conditions for the existence of tight designs suggest that we should focus on the branches which have growth rate of the form $v=ak+b+o(1)$ with $a\geq 2$. Figure 5.1 indicates that it is likely that there is only one such branch but we do not need this fact here.

Lemma 5.2. *If there exists a function $\widetilde{v}(k):\mathbb{R}^+\to\mathbb{R}^+$ such $f_4(\widetilde{v}(k),k)=0$ and $\widetilde{v}(k)\geq 2k$ for all $k\in\mathbb{R}^+$, and*

$$\widetilde{v}(k)=ak+b+o(1)$$

as $k\to+\infty$, then

$$\begin{cases} a=\frac{2}{1-\sqrt[4]{\frac{3}{8}}}=\frac{2}{5}(8+2\sqrt{6}+\sqrt{48+22\sqrt{6}})\approx 9.1971905725,\\ b=\frac{23}{500}\left(249+86\sqrt{6}+\sqrt{171312+70918\sqrt{6}}\right)\approx 48.1640392521. \end{cases} \tag{5.1}$$

Proof. Since $\widetilde{v}(k) \geq 2k$, we have $a \geq 2$. Substituting $\widetilde{v}(k) = ak + b + o(1)$ into $f_4(\widetilde{v}(k), k)$, we obtain

$$\begin{aligned} 0 = f_4(\widetilde{v}(k), k) = &- 128a(1-a)^4(128 - 256a + 192a^2 - 64a^3 + 5a^4)k^{13} \\ &\Big(16(1-a)^2(4096 - 16384a + 12288a^2 + 20480a^3 \\ &- 40232a^4 + 27216a^5 - 7976a^6 + 512a^7 + 21a^8) \\ &- 128(1-a)^3(128 - 1152a + 2112a^2 - 1600a^3 \\ &+ 537a^4 - 45a^5)b\Big)k^{12} \\ &+ o(k^{12}) \end{aligned}$$

as $k \to +\infty$. Therefore, the coefficients of k^{13} and k^{12} on the right-hand side must vanish, which gives us the unique solution in Eq. (5.1). ☐

Proposition 5.3. *The real zeros (v, k) of the polynomial f_4 in the region $k \geq 10^5$ and $2k \leq v \leq \frac{4}{5}k^2$, satisfy*

$$ak + b \leq v \leq ak + b + \frac{1}{100},$$

where a and b are the numbers in Eq. (5.1).

The proof is given in §B.

6 Proof of Theorem 1.2

Assume that there exists a nontrivial tight 8-design. By the construction of complementary designs, we know that there must then exist a nontrivial tight 8-(v_0, k_0, λ) design with parameters $v_0 \geq 2k_0$ and $k_0 \geq 8$. Furthermore, by Proposition 5.1, $f_4(v_0, k_0) = 0$ holds. A computer check shows that the polynomial $f_4(v, k)$ has no integer zeros in the region $8 \leq k \leq 10^5$ and $v \geq 2k$, so $k_0 \geq 10^5$.

Recall from Remark 4.2 and Proposition 4.3 that

$$q_1 = \frac{4(k-4)(k-3)}{v} \in \mathbb{Q}(v, k),$$

and $q_1(v_0, k_0)$ is a positive integer. If $q_1(v_0, k_0) \in \{1, 2, 3, 4\}$, then substituting $v_0 = \frac{4(k_0-4)(k_0-3)}{q_1(v_0,k_0)}$ into $f_4(v_0, k_0) = 0$, we get $k \in \{2, 3, 4\}$, which is too small since $k_0 \geq 10^5$. Thus, $q_1(v_0, k_0) \geq 5$, which implies that $v_0 \leq \frac{4}{5}k_0^2$. Let

$$v_- := ak + b \quad \text{and} \quad v_+ := ak + b + \frac{1}{100},$$

where a, b are given in (5.1). Applying Proposition 5.3, we know that $v_- \leq v_0 \leq v_+$.

Let

$$X := 48q_4 - 16q_3 + 6q_2 - 144q_1 + 45q_0 \in \mathbb{Q}(v, k),$$

where $q_0, \ldots, q_4 \in Q(v, k)$ are given in Remark 4.2. The rational function X is an integral linear combination of $q_0, \ldots, q_4$. So, the number $X(v_0, k_0)$ is an integral linear combination of $q_0(v_0, k_0), \ldots, q_4(v_0, k_0)$, which are all integers by Proposition 4.3. Thus, $X(v_0, k_0)$ is an integer as well.

Expanding now the expression of $X(v_0, k_0)$, we obtain

$$X(v_0, k_0) = \frac{9(k_0-4)}{(v_0-7)^4}(g_0 - g_1v_0 + g_2v_0^2 - g_3v_0^3 + g_4v_0^4),$$

where

$$
\begin{aligned}
g_0 &:= 128k_0^4 + 256k_0^3 - 896k_0^2 - 1024k_0 - 1944, \\
g_1 &:= 256k_0^3 + 192k_0^2 - 1088k_0 - 2186, \\
g_2 &:= 192k_0^2 - 833, \\
g_3 &:= 64k_0 - 82, \\
g_4 &:= 5,
\end{aligned}
$$

and find that $g_0, \dots, g_4 \geq 0$ since $k_0 \geq 10^5$. Therefore,

$$
\begin{aligned}
X(v_0, k_0) &\geq \frac{9(k_0-4)}{(v_+ - 7)^4}\left(g_0 - g_1 v_+ + g_2 v_-^2 - g_3 v_+^3 + g_4 v_-^4\right), \\
X(v_0, k_0) &\leq \frac{9(k_0-4)}{(v_- - 7)^4}\left(g_0 - g_1 v_- + g_2 v_+^2 - g_3 v_-^3 + g_4 v_+^4\right).
\end{aligned}
$$

The right-hand sides of the above equations are single variable rational functions in k_0. So, it is easy to show that when $k_0 \geq 10^5$,

$$235 + \frac{1}{4} \leq X(v_0, k_0) \leq 235 + \frac{3}{4},$$

which contradicts the fact that $X(v_0, k_0)$ is an integer. $\square$

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A The thirteen degree polynomial

$f_4(v,k) := -16384k^{12}v + 65536k^{12} + 98304k^{11}v^2 - 393216k^{11}v - 253952k^{10}v^3 + 786432k^{10}v^2 + 1744896k^{10}v - 3309568k^{10} + 368640k^9v^4 - 327680k^9v^3 - 8724480k^9v^2 + 16547840k^9v - 328320k^8v^5 - 1102464k^8v^4 + 17194752k^8v^3 - 21567744k^8v^2 - 49810560k^8v + 62323584k^8 + 182784k^7v^6 + 2050560k^7v^5 - 16432128k^7v^4 - 13016064k^7v^3 + 199242240k^7v^2 - 249294336k^7v - 61184k^6v^7 - 1642240k^6v^6 + 6536960k^6v^5 + 58253568k^6v^4 - 293538048k^6v^3 + 209662720k^6v^2 + 511604992k^6v - 488998144k^6 + 10752k^5v^8 + 698880k^5v^7 + 1258752k^5v^6 - 59703552k^5v^5 + 183266304k^5v^4 + 243542016k^5v^3 - 1534814976k^5v^2 + 1466994432k^5v - 640k^4v^9 - 143664k^4v^8 - 2296192k^4v^7 + 27050224k^4v^6 - 7038496k^4v^5 - 582955856k^4v^4 + 1856597696k^4v^3 - 1428764528k^4v^2 - 1015706784k^4v + 974873344k^4 + 7520k^3v^9 + 772608k^3v^8 - 2875616k^3v^7 - 58917568k^3v^6 + 469164960k^3v^5 - 1155170432k^3v^4 + 412538336k^3v^3 + 2031413568k^3v^2 - 1949746688k^3v + 336k^2v^{10} - 52816k^2v^9 - 1582560k^2v^8 + 27560816k^2v^7 - 127930016k^2v^6 + 28759472k^2v^5 + 1497511456k^2v^4 - 4944873072k^2v^3 + 6922441360k^2v^2 - 4733985888k^2v + 1506333312k^2 - 2352kv^{10} + 203472kv^9 - 764688kv^8 - 24513072kv^7 + 293023248kv^6 - 1459281552kv^5 + 3929166288kv^4 - 5947568016kv^3 + 4733985888kv^2 - 1506333312kv + 45v^{11} + 972v^{10} - 191952v^9 + 2961396v^8 - 14780538v^7 - 18769932v^6 + 544096980v^5 - 2755473732v^4 + 7281931941v^3 - 11097146016v^2 + 9310949028v - 3408102864.$

B Proof of Proposition 5.3

Lemmas B.1, B.2, B.3, and B.4, prove Proposition 5.3. The proofs of these lemmas are not difficult once the appropriate auxiliary polynomials g has been explicitly identified. Such auxiliary polynomials were obtained through an ad-hoc procedure that we will not describe here.

Lemma B.1. *If $v \in [2k, 9k]$ and $k \geq 20000$, then $f_4(v,k) > 0$.*

Proof. Let

$$\begin{aligned} g(v,k) :=& 65535k^{12} - 16404k^{12}v + 98304k^{11}v^2 - 253969k^{10}v^3 + \\ & 368584k^9v^4 - 328320k^8v^5 + 182701k^7v^6 - 61184k^6v^7 + \\ & 10744k^5v^8 - 640k^4v^9 + 335k^2v^{10} + 45v^{11}. \end{aligned}$$

View $f_4(v,k)$ and $g(v,k)$ as polynomials in v with coefficients in $\mathbb{R}[k]$. Using the fact that $k \geq 20000$, we can check that for every $i \geq 0$, the coefficient of v^i in $f_4(v,k)$ is no smaller than that in $g(v,k)$, and for some i, the coefficient of v^i in $f_4(v,k)$ is strictly larger than that in $g(v,k)$. So, we find that $f_4(v,k) > g(v,k)$ since $v \geq 2k > 0$.

Let $t := v/k$, so that $t \in [2,9]$. We have

$$f_4(v,k) > g(v,k) = g(tk,k) = h_{13}(t)k^{13} + h_{12}(t)k^{12} + h_{11}(t)k^{11},$$

where

$$\begin{aligned} h_{13}(t) :=& -16404t + 98304t^2 - 253969t^3 + 368584t^4 \\ & - 328320t^5 + 182701t^6 - 61184t^7 + 10744t^8 - 640t^9, \\ h_{12}(t) :=& 65535 + 335t^{10}, \\ h_{11}(t) :=& 45t^{11}. \end{aligned}$$

One verifies that h_{13}, h_{12} and h_{11} are positive when $t \in [2,9]$, and then the result follows. □

Recall the definitions of real numbers a and b in Eq. (5.1):

$$\begin{cases} a = \frac{2}{1-\sqrt[4]{\frac{3}{8}}} = \frac{2}{5}(8 + 2\sqrt{6} + \sqrt{48 + 22\sqrt{6}}) \approx 9.1971905725, \\ b = \frac{23}{500}\left(249 + 86\sqrt{6} + \sqrt{171312 + 70918\sqrt{6}}\right) \approx 48.1640392521. \end{cases}$$

Lemma B.2. *If $v \in [9k, ak+b]$ and $k \geq 100$, then $f_4(v,k) > 0$.*

Proof. Let $t := ak + b - v$, so that $t \in [0, (a-9)k + b] \subseteq [0, 0.9k]$ when $k \geq 100$. Let

$$\begin{aligned} g(t,k) :=& 13700000000tk^{12} - 13400000000t^2k^{11} - 140000000t^3k^{10} \\ & - 1380000000t^4k^9 - 6000000t^5k^8 - 21000000t^6k^7 \\ & - 10000t^7k^6 - 73000t^8k^5 - 40000t^9k^3 - 7t^9k^4 \\ & - 50000t^{10}k - 200t^{10}k^2 - 312000000000t^{11}. \end{aligned}$$

View $f_4(ak+b-t,k)$ and $g(t,k)$ as polynomials in k with coefficients in $\mathbb{R}[t]$. Using the fact that $t \geq 0$, we can check that for every $i \geq 0$, the coefficient of k^i in $f_4(ak+b-t,k)$ is no smaller than that in $g(t,k)$, and for some i, the coefficient of k^i in $f_4(ak+b-t,k)$ is strictly larger than that in $g(t,k)$. So, we find that $f_4(ak+b-t,k) > g(t,k)$ since $k > 0$.

Let $s := t/k$, so that $s \in [0, 0.9]$. We have

$$f_4(v,k) = f_4(ak+b-t) > g(t,k) = g(sk,k) = h_{13}(s)k^{13} - h_{12}(s)k^{12} - h_{11}(s)k^{11},$$

where

$$\begin{aligned} h_{13}(s) :=& 13700000000s - 13400000000s^2 - 140000000s^3 \\ & - 1380000000s^4 - 6000000s^5 - 21000000s^6 - 10000s^7 \\ & - 73000s^8 - 7s^9, \\ h_{12}(s) :=& 40000s^9 + 200s^{10}, \\ h_{11}(s) :=& 50000s^{10} + 312000000000s^{11}. \end{aligned}$$

One verifies that h_{13}, h_{12} and h_{11} are positive when $s \in [0, 0.9]$. Since $k \geq 100$

$$
\begin{aligned}
f_4(v,k) >& h_{13}(s)k^{13} - h_{12}(s)k^{12} - h_{11}(s)k^{11} \geq (h_{13}(s) - h_{12}(s)/100 - h_{11}(s)/100^2)k^{13} \\
=& \Big(13700000000s - 13400000000s^2 - 140000000s^3 \\
& - 1380000000s^4 - 6000000s^5 - 21000000s^6 - 10000s^7 \\
& - 73000s^8 - 407s^9 - 7s^{10} - 31200000s^{11} \Big) k^{13}.
\end{aligned}
$$

The result follows from the fact that the right-hand side of the expression above is positive when $s \in [0, 0.9]$. $\square$

Lemma B.3. *If $ak + b + \frac{1}{100} \leq v \leq 10k$ and $k \geq 10^5$, then $f_4(v,k) < 0$.*

Proof. Let $t := v - ak - b$, so that $t \in [\frac{1}{100}, (10-a)k - b] \subseteq [\frac{1}{100}, k]$ when $k \geq 10^5$. Let

$$
\begin{aligned}
g(t,k) :=& - 137892000k^{12} + 13318642886180k^{11} + 713748202323829k^{10} \\
& + 48837673261525668k^9 + 5098485316801241991k^8 \\
& + 980132640412645508268k^7 + 48891070993530297659493 4k^6 \\
& + 1305009906289977795675277621k^5 \\
& + 151418572251274917743210971453210k^4 \\
& + 64000t^9k^3 + 2000t^{10}k^2 + 7000t^{10}k + 127t^{11}.
\end{aligned}
$$

View $f_4(ak+b+t,k)$ and $g(t,k)$ as polynomials in k with coefficients in $\mathbb{R}[t]$. Using the fact that $t \geq \frac{1}{100}$, we can check that for every $i \geq 0$, the coefficient of k^i in $f_4(ak+b+t,k)$ is no larger than that in $g(t,k)$, and for some i, the coefficient of k^i in $f_4(ak+b+t,k)$ is strictly smaller than that in $g(t,k)$. So, we find that $f_4(ak+b+t,k) < g(t,k)$ since $k > 0$. It follows from $t \leq k$ that $g(t,k) \leq g(k,k)$. It is easy to show the one variable polynomial $g(k,k)$ takes negative value when $k \geq 10^5$. Thus, $f_4(v,k) < g(t,k) \leq g(k,k) < 0$. $\square$

Lemma B.4. *If $v \in [9.24k, 0.8k^2]$ and $k \geq 10^5$, then $f_4(v,k) < 0$.*

Proof. Let

$$
\begin{aligned}
g(v,k) :=& 65536k^{12} - 16384k^{12}v + 98312k^{11}v^2 - 253952k^{10}v^3 \\
& + 368640k^9v^4 - 328299k^8v^5 + 182784k^7v^6 - 61177k^6v^7 \\
& + 10752k^5v^8 - 639k^4v^9 + 336k^2v^{10} + 45v^{11}.
\end{aligned}
$$

View $f_4(v,k)$ and $g(v,k)$ as polynomials in v with coefficients in $\mathbb{R}[k]$. Using the fact that $k \geq 10^5$, we can check that for every $i \geq 0$, the coefficient of v^i in $f_4(v,k)$ is no larger than that in $g(v,k)$, and for some i, the coefficient of v^i in $f_4(v,k)$ is strictly smaller than that in $g(v,k)$. So, we find that $f_4(v,k) < g(v,k)$ since $v > 0$.

Let $t := v/k$, so that $t \in [9.24, 0.8k]$. Let

$$
\begin{aligned}
h(x) :=& - 1 + 16384x - 98312x^2 + 253952x^3 - 368640x^4 \\
& + 328299x^5 - 182784x^6 + 61177x^7 - 10752x^8 + 639x^9.
\end{aligned}
$$

We have

$$
\begin{aligned}
& f_4(v,k) < g(v,k) = g(tk,k) \\
=& - \big(k - 65536\big)k^{12} - \big(h(t) - 298t^9\big)k^{13} \\
& - \big(298 - 336(t/k) - 45(t/k)^2\big)t^9k^{13} \qquad \text{(B.1)} \\
=& - \big(k - 65536\big)k^{12} - \big(h(t) - 0.1t^9\big)k^{13} \\
& - \big(0.1 - 336(t/k) - 45(t/k)^2\big)t^9k^{13}. \qquad \text{(B.2)}
\end{aligned}
$$

It suffices to prove that the coefficients of k^{12}, k^{13} and t^9k^{13} in Eq. (B.1) are all negative, or those coefficients in Eq. (B.2) are all negative. Since $k \geq 10^5$, we have $0.00026k^2 \geq 26k$. So, either $v \geq 26k$ or $v \leq 0.00026k^2$.

Case *1*: $v \in [26k, 0.8k^2]$, so that $t \in [26, 0.8k]$ and $t/k \in [0, 0.8]$.

Consider Eq. (B.1). The result follows from the following facts:

(i) $k - 65536 \geq 0$;

(ii) $h(t) - 298t^9 \geq 0$ when $t \geq 26$;

(iii) $298 - 336(t/k) - 45(t/k)^2 \geq 0$ when $t/k \in [0, 0.8]$.

Case *2*: $v \in [9.24k, 0.00026k^2]$ so that $t \in [9.24, 0.00026k]$ and $t/k \in [0, 0.00026]$.

Consider Eq. (B.2). The result follows from the following facts:

(i) $k - 65536 \geq 0$;

(ii) $h(t) - 0.1t^9 \geq 0$ when $t \geq 9.24$;

(iii) $0.1 - 336(t/k) - 45(t/k)^2 \geq 0$ when $t/k \in [0, 0.00026]$. ☐

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