INTEGRAL POINTS ON VARIABLE SEPARATED CURVES (DRAFT)

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1. Introduction

Xiang proves the non-existence of non-trivial tight 8-designs in algebraic combinatorics in [27, 1.2]. His method of proof led him to a conjecture in elementary number theory which, if true, could be used to give a new proof of the fact that there exist only finitely non-trivial tight $2e$-designs for $e \geq 4$. His conjecture 4.5 in [27] can be stated as follows.

Conjecture 1.1. (Z. Xiang) *Let $n \geq 3$ be an integer. For every nonzero integer c, there are only finitely many pairs (x, y) of positive integers satisfying $y \geq x + 2$ and such that the n rational numbers*

$$\frac{x(x+1)}{y}, \frac{x(x+1)(x+2)}{y(y+1)}, \ldots, \frac{x(x+1)(x+2)\cdots(x+n)}{y(y+1)\cdots(y+n-1)}$$

all have denominators that divide c.

We show in 3.7 that Conjecture 1.1 can be reduced to a statement pertaining to only one of the rational functions appearing in 1.1. Namely, we prove that Conjecture 1.1 follows from Conjecture 1.2 below.

Conjecture 1.2. (a) *Fix a positive integer c, $c \neq 1, 4$. Then the equation*

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2) \tag{1.3}$$

has only finitely many solutions in positive integers (x, y, b) with $y \geq x+2$ and $\gcd(b, c) = 1$.

(b) *Let $c = 1$ or $c = 4$. Then Equation* (1.3) *has only finitely many solutions in positive integers (x, y, b) with $y \geq x + 2$ in addition to the infinite family of solutions described in Proposition* 3.5.

Equations of the form (1.3) have been studied already by several authors. A summary of earlier work on this type of equations when b and c are fixed is found in the introduction of Saradha and Shorey [19], where its study is motivated by a problem of Erdös. The integer solutions to (1.3) have been completely determined for instance when $c = b = 1$ [4], and when $c = 1$ and $b = 4$ [21, Table T34]. We provide in section 2 extensive numerical evidence that Conjecture 1.2 might hold.

We investigate in this article whether there might exist general principles pertaining to a wider class of equations that could be applied to the particular Equation (1.3). For this, we will consider equations of the form $f(x) = bg(y)$ where $f(x) \in \mathbb{Z}[x]$ and $g(y) \in \mathbb{Z}[y]$

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have positive degree, and ask whether conditions can be given on x and y to insure that on a certain region R of the (x, y)-plane, the equation $f(x) = bg(y)$ only has finitely many integer solutions (x_0, y_0, b_0) with $(x_0, y_0) \in R$. The diophantine properties of equations of the type $f(x) = g(y)$ is a subject with a long history, and we refer to [2] for some relevant literature. A first such region R is easily obtained in our next proposition, proved in 4.1. Let $D := \frac{\deg(f)}{\deg(g)}$.

Proposition 1.4. *Let $f(x) \in \mathbb{Q}[x]$ and $g(y) \in \mathbb{Q}[y]$ be of positive degree. Let $\delta > 0$. Then there exist only finitely many solutions (x_0, y_0, b_0) in positive integers to the equation $f(x) = bg(y)$ such that $y_0 \geq x_0^{D+\delta}$. This set of solutions is effectively computable.*

Let us recall here some geometric facts that will allow us to strengthen the above proposition. Let $\mathbb{Q}(a)$ denote the field of rational functions in the variable a, with algebraic closure $\overline{\mathbb{Q}(a)}$. Let $f(x) \in \mathbb{Q}[x]$ and $g(y) \in \mathbb{Q}[y]$ be of positive degree and consider the polynomial $f(x) - ag(y) \in \mathbb{Q}(a)[x, y]$. Assume that this polynomial is geometrically irreducible (that is, it is irreducible in $\overline{\mathbb{Q}(a)}[x, y]$. An example of a polynomial which in not irreducible in that ring is $x^n - ay^n$ for $n \geq 2$.) A geometrically irreducible polynomial $f(x) - ag(y)$ defines a unique smooth proper geometrically connected curve $X/\mathbb{Q}(a)$. We will always assume in this article that the genus of X is positive. Then there exists only finitely many rational numbers b such that the polynomial $f(x) - bg(y)$ is either not geometrically irreducible, or is geometrically irreducible but the associated smooth proper geometrically connected curve $X_b/\mathbb{Q}$ is not of genus equal to the genus of X.

Example 1.5 (a) Fix an integer c, $c \neq 0$. Then the polynomial

$$cx(x+1)(x+2)(x+3) - ay(y+1)(y+2) \tag{1.6}$$

defines a smooth proper geometrically connected curve of genus 3 over $\mathbb{Q}(a)$. When b is a non-zero integer, the equation

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2) \tag{1.7}$$

defines a smooth proper geometrically connected curve of genus 3 over $\mathbb{Q}$.

(b) The polynomial

$$x(x+1)(x+2)(x+3) - ay(y+1) \tag{1.8}$$

defines a smooth proper geometrically connected curve of genus 1 over $\mathbb{Q}(a)$. The curve has two involutions given by $x \mapsto -x-3$ and $y \mapsto -y-1$. The Jacobian of X has j-invariant equal to $\frac{(a+13/3)^3}{(a-4)^2(a+9/4)}$. The equation

$$x(x+1)(x+2)(x+3) = by(y+1) \tag{1.9}$$

with $b = 4$ defines an affine plane curve with a singularity at $(\alpha, -1/2)$ with $\alpha^2 + 3\alpha + 1 = 0$, whose smooth projective model $X_{b=4}/\mathbb{Q}$ has genus 0. The affine curve can be parameterized by $x(t) = t$ and $y(t) = t(t+3)/2$. In particular, this affine curve has infinitely many integer points. When $b = 12$, the genus of $X_{b=12}$ is 1, and the integer solutions of (1.9) are described in [21, Theorem A24]. An upperbound for $\max(|x|, |y|)$ in terms of b for an integer solution

(x, y) of (1.9) can be obtained from Runge's method (see [22, Theorem, page 186]). We will return to Equation (1.9) in section 6.

Proposition 1.10. *Let $f(x) \in \mathbb{Q}[x]$ and $g(y) \in \mathbb{Q}[y]$ be of positive degree. Assume that the polynomial $f(x) - ag(y) = 0$ defines a smooth proper geometrically connected curve $X/\mathbb{Q}(a)$ of positive genus. Then there exist only finitely many solutions (x_0, y_0, b_0) in positive integers to the equation $f(x) = bg(y)$ such that the associated smooth proper geometrically connected curve X_{b_0} has positive genus and such that $y_0 \geq x_0^D$.*

This proposition, proved in 4.2, follows from Siegel's Theorem on integer points on affine curves ([9, D.9.2.2]). We investigate in this article whether Proposition 1.10 could be strengthened to state that there exists $\epsilon > 0$, depending on f and g, such that there there exist only finitely many solutions (x_0, y_0, b_0) in positive integers to the equation $f(x) = bg(y)$ such that the associated smooth projective curve X_{b_0} has positive genus and such that $y_0 \geq x_0^{D-\epsilon}$. Our main result on this question is Proposition 4.3, whose proof assumes that the *abc*-conjecture holds.

1.11 In Conjecture 1.2, the equation $f(x) = bg(y)$ is such that $D = 4/3$. The conjecture asserts that when $c \neq 1, 4$, there exists only finitely many solutions (x_0, y_0, b_0) in positive integers such that $y \geq x + 2$. There exist infinitely many solutions (x_0, y_0, b_0) in positive integers such that $y_0 = x_0 + 1$, due to the following parametric family of solutions: $x(t) = t$, $y(t) = t + 1$, and $b(t) = t$. More generally, let $f(x) \in \mathbb{Z}[x]$ and $g(y) \in \mathbb{Z}[y]$ have positive degree, and suppose that we can find $x(t), y(t), b(t) \in \mathbb{Q}[t]$ such that the equation

$$f(x(t)) = b(t)g(y(t))$$

holds in $\mathbb{Q}[t]$. Clearly when $x(t)$ is not constant,

$$\frac{\deg_t(y)}{\deg_t(x)} = \frac{\deg(f)}{\deg(g)} - \frac{\deg_t(b)}{\deg(g)\deg_t(x)} \leq \frac{\deg(f)}{\deg(g)} =: D. \tag{1.12}$$

Let $\mathrm{Int}(\mathbb{Z})$ denote the ring of integer-valued polynomials (that is, polynomials $h(t) \in \mathbb{Q}[t]$ with $h(\mathbb{Z}) \subseteq \mathbb{Z}$.)

Proposition 1.13. *Let $f(x) \in \mathbb{Z}[x]$ and $g(y) \in \mathbb{Z}[y]$ have positive degree. Assume that $f(x)-ag(y)$ is geometrically irreducible in $\mathbb{Q}(a)[x, y]$, and that its associated smooth projective curve over $\mathbb{Q}(a)$ has positive genus. Suppose that we can find a solution as above with $x(t), y(t), b(t) \in \mathrm{Int}(\mathbb{Z})$ and both $x(t)$ and $b(t)$ not constant. Assume in addition that the leading coefficients of $x(t)$, $y(t)$, and $b(t)$ have positive leading coefficients. If there exists $\epsilon > 0$ such that there exist only finitely many solutions (x_0, y_0, b_0) in positive integers to the equation $f(x) = bg(y)$ such that $y_0 \geq x_0^{D-\epsilon}$, then*

$$\frac{\deg_t(y)}{\deg_t(x)} < D - \epsilon.$$

Consider then the set

$$\mathcal{D} := \left\{ \frac{\deg_t(y)}{\deg_t(x)}, \text{ with } x(t), y(t), b(t) \text{ as in 1.13} \right\}.$$

As noted in (1.12), $\mathcal{D}$ is bounded above by D. Any reduced fraction β/α with $\beta/\alpha < D$ could possibly belong to $\mathcal{D}$. Indeed, this would happen if a solution with $\deg_t(x) = \alpha$, $\deg_t(y) = \beta$, and $\deg_t(b) = D\alpha - \beta$ could be found. The only known restriction comes from Mordell's Conjecture over function fields (see, e.g., [18] or [25]), which, when applicable, implies that the set $\mathcal{D}$ can only contain at most finitely many fractions with $D\alpha - \beta = 1$. On the other hand, in the context of Conjecture 1.2 (a), if this conjecture holds, then the corresponding set $\mathcal{D}$ does not intersect the region $(1, \infty)$, and in particular, in this case we expect $\sup \mathcal{D} < D = 4/3$. We investigate in section 5 whether the strict inequality $\sup \mathcal{D} < D$ might hold more generally. Among other results, we show the following theorem. Recall that when F is any field and $g(t) \in F[t]$, then the *radical* $\mathrm{rad}(g)$ is defined to be the product of the distinct irreducible factors of g.

Theorem 1.14. (See Theorem 5.8.) *Let $f(x) \in \mathbb{Q}[x]$ and $g(y) \in \mathbb{Q}[y]$. Let $(x_0(t), y_0(t), b_0(t))$ be a solution of the equation $f(x) = bg(y)$ with $x_0(t), y_0(t), b_0(t) \in \mathbb{C}[t]$ and $x_0(t)$ not constant. If g has at least one multiple root, and $\deg(f) > \deg(g)$ and f has no multiple roots, then $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{1}{\deg(g) - \deg(\mathrm{rad}(g))} < \frac{\deg(f)}{\deg(g)}$.*

There are about 369,000 positive integer solutions (x, y, b, c) to Equation (1.9), with $c \in [1, 300]$, $x \in [1, 10^9]$, $\gcd(b, c) = 1$, $b \neq 4c$ and $\frac{\log(y)}{\log(x)} \geq 4/3$. In section 6, we remove from this set of solutions all solutions which can be explained 'geometrically', that is, solutions which we found to belong to a parametric family, and introduce a counting function $N(B)$ for the remaining set of 'not-geometrically explained' solutions. The data in 6.7 shows a surprisingly good fit between $N(B)$ and a function of the form $\alpha - \beta e^{-\gamma B}$, where α, β, γ are positive constants. It would be interesting to provide a heuristic or theoretical argument that would explain this fit.

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2. Numerical evidence

Our goal in this section is to provide some numerical evidence that Conjecture 1.2 (a) holds. Let $f(x) := x(x+1)(x+2)(x+3)$ and $g(x) := x(x+1)(x+2)$. Fix a positive integer c. Consider the diophantine equation (1.3) $cf(x) = bg(y)$ in the variables (x, y, b):

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2).$$

When $c = 1$, we find two obvious parametric solutions $(x(t), y(t), b(t))$ with integer polynomials having positive leading coefficients, namely $(t-3, t-3, t)$ and $(t, t+1, t)$. When $c = 1$ or $c = 4$, Equation (1.3) also admits a parametric family of solutions of a different kind, with $y \geq x + 2$, and these cases are treated in the next section.

We computed the set of all solutions (x, y, b, c) of (1.3) with $c \in [1, 300]$, $y \geq x+2$, $x \leq 10^9$ and $\gcd(b, c) = 1$. We found 5050 such solutions. The data is available for download on the website of the senior author. We report now on this data, which supports the above conjecture.

Remark 2.1 The determination of all solutions to (1.3) for a given c is computationally very expensive already when the upper bound for x is 10^9. We used 150 CPU cores on the zcluster at the University of Georgia to produce our data. The computation of all solutions with $x \leq 10^9$ was most expensive in the case of the prime $c = 179$, and took 82.45 days on a single core. The case $c = 1$ took 80.45 days on a single core. The shortest case was $c = 210$, where the computation took 1.93 days. Our algorithm runs as follows.

(1) Fix c, and for each x in the chosen domain, list all divisors b of $f(x)$ such that $\gcd(b, c) = 1$.

(2) For each b found for a given x, check the existence of y such that $cf(x) = bg(y)$. This implies computing the $\deg(g)$-root of $cf(x)/b$ to some low precision, and then checking the value $g(y_0)$ at a few integers y_0 close to this root.

In step (1), instead of listing all divisors b of $f(x)$ and then checking that $\gcd(b, c) = 1$, we list all divisors of

$$\frac{x}{\gcd(x,c)} \cdot \frac{(x+1)}{\gcd(x+1,c)} \cdot \frac{(x+2)}{\gcd(x+2,c)} \cdot \frac{(x+3)}{\gcd(x+3,c)}.$$

Proceeding in this manner is much faster if c has many (small) prime factors, as in the case of $c = 210$ mentioned above.

In the table below, for each value of c, we exhibit the solution (x, y, b) to (1.3) with $y \geq x + 2$ which has the *largest value of* x among the *complete* list of all such solutions with $x \leq 10^9$ and $\gcd(b, c) = 1$. As we shall see, this largest x-value remains quite small compared to 10^9. We indicate also for a given c the number n of solutions to (1.3) with $y \geq x + 2$ and $0 < x \leq 10^9$.

c	x	y	b	n	c	x	y	b	n
2	284	1064	11	9	**22**	260	262	5655	12
3	713	1610	187	10	**23**	59943	61594	1270834	41
5	285	350	779	19	**24**	284	286	6745	5
6	68	70	391	4	**25**	351	648	1412	12
7	9590	59730	278	17	**26**	635	1014	4081	20
8	142	638	13	5	**27**	320	322	8560	6
9	104	106	910	6	**28**	3924	20383	785	11
10	207	350	437	7	**29**	16352	29783	78504	34

c	x	y	b	n	c	x	y	b	n
11	1918	4520	1616	33	**30**	356	358	10591	5
12	140	142	1645	2	**31**	894	1610	4768	23
13	358015	564718	1185928	32	**32**	380	382	12065	2
14	779	1064	4301	14	**33**	15040	33462	45080	19
15	176	178	2596	6	**34**	620	712	13995	19
16	713	895	5797	5	**35**	1935	2924	19668	29
17	10582	37960	3899	34	**36**	10945	17710	93041	3
18	272	350	2329	7	**37**	8225	20349	20108	32
19	4030	9918	5143	37	**38**	18422	29279	174405	14
20	7565	25024	4183	7	**39**	197583	260494	3362621	20
21	6498	7370	93575	18	**40**	476	478	18921	6

Consider the parametric solution (x, y, b, c) to (1.3) with[1]

$$x(c) := 12c - 4, \quad y(c) := 12c - 2, \quad b(c) := (12c - 7)c + 1. \tag{2.2}$$

Among the solutions (x, y, b, c) with $c \in [1, 300]$, $y \geq x + 2$, and $x \leq 10^9$, there are 83 values of c, including $c = 6, 9, 12, 15, 22, 24, 27, 30, 32, 40$ in the above table, where for such c the solution (x, y, b, c) with largest x-value is equal to the solution (2.2). In particular, for these c-values, since $x = 12c - 4$, the 'largest' solution found when c is fixed, has a very small x-value. There are an additional 73 values of c where exactly one solution (x, y, b, c) has $y > \max(x + 2, 12c - 4)$.

In the complete list of all 5050 solutions to (1.3) with $c \in [1, 300]$, $y \geq x + 2$, $x \leq 10^9$, and $\gcd(b, c) = 1$, only 55 solutions (x, y, b, c) have $x \in [10^5, 10^9]$, and only 21 solutions have $x \in [10^6, 10^9]$. We list below the values c for which these 21 solutions with $x > 10^6$ occur:

$$1, 4, 44, 49, 65, 79, 89, 104, 139, 156, 161, 185, 223, 263, 298.$$

Five of these large solutions occur with $c = 1$ or 4, and as discussed in the next section, these solutions in fact belong to an infinite family. Of the 16 remaining solutions with $c \neq 1, 4$, all have $x \in [10^6, 10^7]$, except when $c = 79$ and 156, where $x \in [10^7, 10^8]$. The fact that we found only a very small number of solutions with $x \in [10^6, 10^9]$, and that most of these large x-values are close to 10^6, provides some evidence that Conjecture 1.2 may hold. We list below the two solutions with $x > 10^7$ found when $c \neq 1, 4$. In case of $c = 79$, we found

[1]A slighly more general parametrization is as follows: $x(c) := \beta c - 4$, $y(c) = \beta c - 2$, and $b(c) = (\beta c - 7)c + 12/\beta$, with β a positive integer dividing 12. When c is prime to 6, such parametrization produces a solution in positive integers with $\gcd(b, c) = 1$.

two solutions with $x > 10^6$. This also occurred for $c = 89$ and $c = 263$.

c	x	y	b
79	1509512	36635458	8342
79	35125944	36635458	2445875082
156	75128030	122201728	2723331973

Remark 2.3 For the value $c = 187$, we found 69 solutions (x, y, b) in positive integers with $y > x + 1$ and $x \leq 10^9$. For the values $c = 32, 96, 108, 192, 200, 240, 252$, the only solutions to Equation (1.3) in positive integers with $y > x + 1$ and $x \leq 10^9$ have $y = x + 2$. It is natural to wonder whether there exist infinitely many values of c such that the only solutions (x, y, b, c) in positive integers to Equation (1.3) with $y > x + 1$ have $y = x + 2$. We leave it as an exercise to show that for a given c and for a given $e \geq 2$, there are only finitely many solutions in positive integers to (1.3) with $y = x + e$.

Remark 2.4 For $c = 1$ and a fixed integer b, the set of integer solutions to the equation (1.3) can be completely determined numerically when b is not too large. Indeed, the quotient by the involution $x \mapsto -x - 3$ is an elliptic curve given by the affine equation $X(X + 2) = by(y + 1)(y + 2)$, with $X := x(x + 3)$. Setting $v := b(X + 1)$ and $u := b(y + 1)$, we find an integral equation for the quotient curve of the form $v^2 = u^3 - b^2u + b^2$.

One can obtain the list of all integral points on this elliptic curve E using the function E.integral_points() in Sage [17]. It is then an easy matter to check what the integral solutions of (1.3) are. When $b = 1$, the set of integral solutions was completely determined in [4]. When $b = 4$, the set of integral solutions is considered in [21, Table T34].

When both b and c are fixed, the equation (1.3) defines a curve of genus 3. The function PointSearch in Magma [3] can be used to find rational solutions. We only found two values of b/c where the associated curve has more than 37 points. When $b/c = 243/182$, Magma finds 44 rational points. When $b/c = 247/7$, Magma finds 43 rational solutions, three of them integral with $y > x$: $(38, 40)$, $(75, 98)$, and $(492, 1188)$.

3. The cases $c = 1$ and $c = 4$.

Letting $a := b/c$, we call $X_a/\mathbb{Q}$ the plane curve given by

$$x(x + 1)(x + 2)(x + 3) = ay(y + 1)(y + 2).$$

This curve has 12 obvious points, namely:

$$(x_0, y_0) \text{ with } x_0 \in \{0, -1, -2, -3\} \text{ and } y_0 \in \{0, -1, -2\}.$$

We discuss in this section how a special geometric fact about the curve $X_a/\mathbb{Q}$ affects its arithmetic. Recall that given any five distinct points in the plane, no three on the same line, there exists a unique smooth conic which contains them, and that in general six points are not all contained on a single conic. It turns out that the twelve obvious points on the curve $X_a/\mathbb{Q}$ can be partitioned in two disjoint packets of six points, each lying on a smooth conic.

More precisely,

$$(-3,0),\ (-2,-2),\ (-2,-1),\ (-1,-1),\ (-1,0),$$

and $(0,-2)$, lie on the conic

$$2x^2 + 2xy - y^2 + 8x + y + 6 = 0, \tag{3.1}$$

and the complement $(-3,-2)$, $(-3,-1)$, $(-2,0)$, $(-1,-2)$, $(0,-1)$, and $(0,0)$, lie on the conic $2x^2-2xy+3y^2+4x+3y = 0$. The real solutions of this latter conic form an ellipse in the plane, and will not be of interest in this discussion. The real solutions of the first conic form a hyperbola, and we find on it infinitely many integral points.

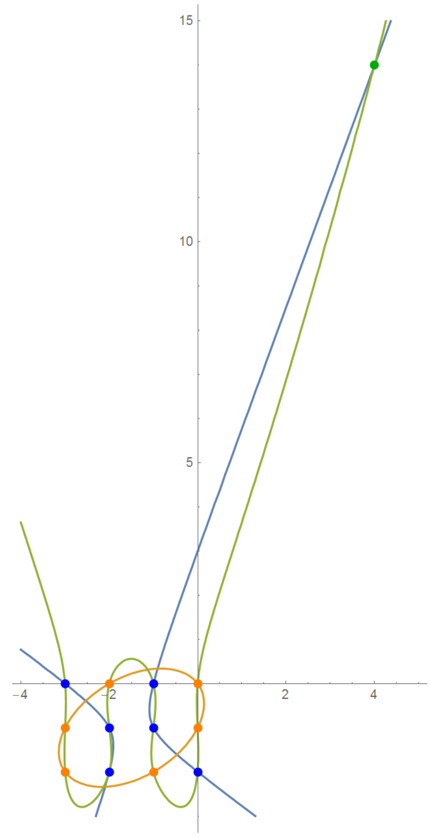
Fix a. Then the intersection of the curve X_a and the conic (3.1) contains the six obvious points listed above plus (at most) two more points (x, y) given by

$$2x = 3(y-3)/4 - a,$$

and

$$(y+1)^2 - 56(y+1)a + 16a^2 - 16 = 0. \tag{3.2}$$

The picture on the right describes the case $a = 1/4$, where the intersection of the curve $X_{a=1/4}$ and the conic (3.1) contains only one additional point, $(4, 14)$, because the two curves share a common tangent line at the point $(-2,-2)$.



The equations were obtained using Magma [3] by asking for the primary decomposition of the ideal (0) in the affine algebra $\mathbb{Q}[x, y, a]/I$, where I is generated by the equation of the curve X_a and the equation of the conic (3.1). Note that substituting $a = \frac{3(y-3)}{4} - 2x$ in (3.2) gives the relation $32(2x^2 + 2xy - y^2 + 8x + y + 6) = 0$.

Setting $a = \frac{3(y-3)}{4} - 2x$, we find that

$$x(x+1)(x+2)(x+3) - ay(y+1)(y+2) =$$
$$(-1/4)(2x^2 + 2xy - y^2 + 8x + y + 6)(2x^2 - 2xy + 3y^2 + 4x + 3y).$$

Therefore, any point (x, y) on the conic (3.1) lies on the curve X_a with $a = \frac{3(y-3)}{4} - 2x$.

Lemma 3.3. *If (x, y) is an integer point on the conic (3.1), then $a = \frac{3(y-3)}{4} - 2x$ is either an integer if y is odd, or its denominator is equal to 4 if y is even.*

Proof. Clearly, if x and y are integers, then $a = b/2^k$ for some $k = 0, 1$ or 2 and odd integer b; and $k = 2$ when y is even. Suppose then that $y = 2t + 1$ for some integer t. Then it follows from (3.1) that

$$x(x + y) + 4x + 3 + y(1 - y)/2 = 0.$$

Since $x(x + y) + 4x$ is always even, we find that $(1 - y)/2$ must be odd. Hence, $y = 4s - 1$ for some integer s, and $(y - 3)/4$ is an integer. □

Let us write the conic (3.1) in standard form:

$$\begin{aligned} 2x^2+2xy-y^2+8x+y+6 &= 3x^2+9x-(y-x)^2+(y-x)+6 \\ &= 3(x+3/2)^2-(y-x-1/2)^2-1/2. \end{aligned}$$

Thus, setting $X = 2y-2x-1$, and $Y := 2x+3$, we find that

$$X^2 - 3Y^2 = -2. \tag{3.4}$$

In particular, any integer point on the conic (3.1) produces an integer solution to $X^2-3Y^2 = -2$.

The integer solutions to (3.4) are well-understood. First, we have an obvious solution $(1,1)$. The fundamental solutions (X,Y) of the Pell equation $X^2-3Y^2=1$ are $(\pm 2, \pm 1)$. Set $\epsilon := 2+\sqrt{3}$. All solutions of $X^2-3Y^2=-2$ are of the form (X,Y) with $X+Y\sqrt{3} = (1+\sqrt{3})\epsilon^i$ for some i (see, e.g., [8, Theorem 3, A] and [20]).

Proposition 3.5. *When $c=1$ or 4, Equation (1.3) has an infinite family of solutions (x,y,b) in positive integers with $y \geq x+2$. More precisely, for each integer $i \geq 2$, given $X_i + Y_i\sqrt{3} := (1+\sqrt{3})\epsilon^i$, consider the point (x_i, y_i) on the conic (3.1) with $x_i := (Y_i-3)/2$ and $y_i := x_i + (X_i+1)/2$. If i is odd, then $(x_i, y_i, b_i := \frac{3(y_i-3)}{4} - 2x_i, c = 1)$ is a solution to (1.3) in positive integers. If i is even, then $(x_i, y_i, b_i := 3(y_i-3) - 8x_i, c=4)$ is a solution to (1.3) in positive integers.*

Proof. Let us note first that x_i and y_i are always integers. For this, it suffices to note that X_i and Y_i are always odd. This is indeed the case because if $X + Y\sqrt{3} = (m+n\sqrt{3})\epsilon$ with m and n odd, then X and Y are always odd.

Since the coefficients of ϵ are positive, it is clear that $X_i, Y_i > 0$. From the formula $y_i = x_i + (X_i+1)/2$, we find that $y_i > x_i + 2$.

To conclude the proof, it remains to show, in view of Lemma 3.3, that i is odd if and only if y_i is odd. It is clear that y_i is odd if and only if $X_i + Y_i$ is divisible by 4. Computing now $X_i + Y_i\sqrt{3} = (m+n\sqrt{3})\epsilon^2$, we find that $X_i = 7m+12n$, and $Y_i = 4m+7n$. In particular, $X_i + Y_i \equiv m+n \mod 4$. When $i=2$, and $m=n=1$, we find that $m+n \equiv 2 \mod 4$. When $i=3$, and $(m,n) = (5,3)$, we find that $m+n \equiv 4 \mod 4$, as desired. $\square$

3.6 Our initial small computer search for solutions (x,y,b) to (1.3) when $c=1$ and $c=4$ found the first four solutions in the infinite sequences given in 3.5 (listed in the table below), and produced the sequence $1, 14, 195, 2716$ for b when $c=1$. This was recognized as Sequence A007655 in The Online Encyclopedia of Integer Sequences [16], and clearly indicated some possible structure in the set of solutions. The analogue sequence for $c=4$ is A028230.

$c=1$	$c=4$
$(19,55,1)$	$(4,14,1)$
$(284,779,14)$	$(75,208,15)$
$(3975,10863,195)$	$(1064,2910,209)$
$(55384,151315,2716)$	$(14839,40544,2911)$

A much larger computer search found all solutions to (1.3) with $c = 1$ or $c = 4$ and $0 \leq x \leq 10^9$ and $y \geq x + 2$. When $c = 1$, only three such solutions, $(2, 4, 10)$, $(8, 10, 6)$, and $(152, 340, 14)$, do not belong to the infinite family in Proposition 3.5. When $c = 4$, we found six exceptional solutions not belonging to the infinite family in Proposition 3.5: $(12, 14, 39)$, $(39, 63, 41)$, $(44, 46, 165)$, $(74, 110, 95)$, $(130, 208, 131)$, and $(5642, 7903, 8217)$. These computations support Conjecture 1.2 (b). We now show that Xiang's Conjecture 1.1 follows from Conjecture 1.2.

Proposition 3.7. *Conjecture* 1.2 *implies Conjecture* 1.1.

Proof. Conjecture 1.1 follows if we can show that for every non-zero positive integer c, there are only finitely many pairs (x, y) of positive integers satisfying $y \geq x + 2$ and such that the denominators of both $\frac{x(x+1)}{y}$ and $\frac{x(x+1)(x+2)(x+3)}{y(y+1)(y+2)}$ divide c. Conjecture 1.2 (a) in this article immediately implies that for every non-zero positive integer c with $c \neq 1, 4$, there are only finitely many pairs (x, y) of positive integers satisfying $y \geq x + 2$ such that the denominator of $\frac{x(x+1)(x+2)(x+3)}{y(y+1)(y+2)}$ divides c.

Assume now that $c = 1$ or 4. Recall that the parametric solutions (x_n, y_n, b_n) of the equation

$$cx(x + 1)(x + 2)(x + 3) = by(y + 1)(y + 2)$$

found in Proposition 3.5 lie on the conic (3.1), which we can rewrite as

$$2(x + 1)(x + 3) - y(y - 2x - 1) = 0.$$

Hence, setting $z_n := y_n - 2x_n - 1$, we have $y_n z_n = 2(x_n + 1)(x_n + 3)$. Using this identity, we get

$$\frac{2x_n(x_n + 1)}{y_n} = z_n - \frac{6(x_n + 1)}{y_n}.$$

For n positive, one checks that $0 < b_n < x_n$, so that

$$\frac{8}{3}(x_n + 1) < y_n < 3(x_n + 1).$$

Hence, $6(x_n + 1)/y_n$ is not an integer and, therefore, $x_n(x_n + 1)/y_n$ is not an integer. Conjecture 1.2 (b) implies that there are only finitely many solutions (x, y) with $y \geq x + 2$ in addition to the parametric solutions, as desired. □

Remark 3.8 Let us return to the sequence introduced in 3.6 when $c = 4$, namely, $1, 15, 209, 2911, \ldots$ (A028230 in [16]). This sequence is recognized as a subsequence of a larger sequence $1, 4, 15,\ 56, 209, 780,\ 2911, \ldots$ (A001353 in [16]). In the entry for A001353 in [16], Jonathan Vos Post asks whether this sequence ever contains a prime. We note in this remark a factorization of the elements of this sequence that shows that the elements of A001353 are never prime.

Recall that $\epsilon := 2 + \sqrt{3}$. Let $\{s_n\}$ denote the sequence A001353, and let us define s_n to be the coefficient of $\sqrt{3}$ in the element ϵ^n. Define $X_n + Y_n\sqrt{3} := (1 + \sqrt{3})\epsilon^n$. We now consider the equality

$$\epsilon^{2n+1} = (X_n + Y_n\sqrt{3})^2\epsilon(1 + \sqrt{3})^{-2}$$

and compare the $\sqrt{3}$-terms. It is easy to check that $\epsilon(1+\sqrt{3})^{-2} = 1/2$, so we obtain the equality $s_{2n+1} = X_n Y_n$. Since $\{X_n\}$ and $\{Y_n\}$ are increasing sequences, we find that s_{2n+1} is never prime.

We now offer a factorization of s_n of the following form. Set $x_n := (Y_n - 3)/2$ and $y_n := (X_n + Y_n)/2 - 1$. This change of variables already appears in 3.5, and we used already in 3.5 the fact that x_n and y_n are integers. It is very easy to check that $s_{n+1} = y_n + 1$. The point (x_n, y_n) lies on the conic (3.1), which we rewrite in the following form:

$$2(x+1)(x+2) = (y+1)(y-2x-2).$$

It follows that

$$s_{n+1} = y_n + 1 = \frac{2(x_n+1)(x_n+2)}{y_n - 2x_n - 2}.$$

We leave it to the reader to check that $y_n - 2x_n - 2 < x_n$, so that the above factorization of s_{n+1} shows that indeed s_{n+1} is always composite.

Remark 3.9 Fix integers $0 < \alpha < \beta < \gamma$, and $0 < m < n$. Consider the following diophantine equation in the variables (x, y, b, c):

$$cx(x+\alpha)(x+\beta)(x+\gamma) = by(y+m)(y+n). \tag{3.10}$$

The equation has 12 obvious integer solutions (x, y, b, c), which occur for all values of b, c, namely:

$$(x, y, b, c) \text{ with } x \in \{0, -\alpha, -\beta, -\gamma\} \text{ and } y \in \{0, -m, -n\}.$$

Let us now impose a new constraint: that the following set of six points $(-\gamma, 0)$, $(-\beta, -n)$, $(-\beta, -m)$, $(-\alpha, -m)$, $(-\alpha, 0)$, $(0, -n)$ all lie on a conic. Note that no three of the last five points on this list can be colinear.

Lemma 3.11. *The six points above all lie on the same conic C if and only if*

$$\gamma = \beta + \alpha \frac{m}{n-m}.$$

Proof. We write down the conic C which contains the last five points in the list by computing the determinant of the 6×6-matrix whose first row is

$$(x^2, xy, y^2, x, y, 1)$$

and whose subsequent five rows are of the form $(a^2, ab, b^2, a, b, 1)$ for each of the five given points (a, b). It turns out that the first point $(-\gamma, 0)$ belongs to C if and only if $\gamma = \beta + \alpha\frac{m}{n-m}$. □

Note that the above condition on γ is compatible with the hypothesis that $\gamma > \beta$. It is possible to find examples of Equation (3.10) which do not have any obvious involution or parametric solutions with c constant (contrary to Equation (1.3)), but where we can find several values of c where Equation (3.10) has infinitely many solutions with $y > x + \gamma$. For instance, when $c \in \{1, 5, 25\}$, then there are infinitely many solutions (x, y, b) in positive integers with $y > x + 6$ to the equation

$$cx(x+3)(x+4)(x+6) = by(y+2)(y+5).$$

4. Variable separated equations

At the cost of slightly weakening Conjecture 1.2, we may state the following unified conjecture which does not distinguish anymore the cases where $c = 1$ and $c = 4$ in Conjecture 1.2: *Fix a positive integer c. Let $\epsilon < 1/3$. The equation*

$$cx(x+1)(x+2)(x+3) = by(y+1)(y+2)$$

has only finitely many solutions in positive integers (x, y, b) with $y \geq x^{4/3-\epsilon}$ and $\gcd(b, c) = 1$. This conjecture is implied by Conjecture 1.2. Indeed, the positive integer solutions (x, y, b, c) found in Proposition 3.5 when $c = 1$ or $c = 4$ all lie on a hyperbola. It follows that for such solutions $\lim_{x\to\infty} \frac{\log(y)}{\log(x)} = 1$. Thus, for any $\delta > 0$, Conjecture 1.2 (b) predicts only finitely many positive integer solutions with $\frac{\log(y)}{\log(x)} \geq 1 + \delta$.

Our goal in this section is to provide evidence that a similar conjecture might be true for a much wider class of equations $f(x) = bg(y)$. Let us first give the proofs of Propositions 1.4 and 1.10 stated in the introduction. Recall that $D := \deg(f)/\deg(g)$.

4.1 Proof of Proposition 1.4. Assume that there exists an infinite sequence $\{(x_n, y_n, b_n)\}$ of distinct solutions in positive integers to the equation $f(x) = bg(y)$ such that for all n, $y_n \geq x_n^{D+\delta}$. Clearly, $\lim_{n\to\infty} x_n = \infty$. We have for all n sufficiently large:

$$b_n = \frac{|f(x_n)|}{|g(y_n)|} \leq \frac{|f(x_n)|}{|g(x_n^{D+\delta})|}.$$

Since $\lim_{x\to\infty} f(x)/g(x^{D+\delta}) = 0$, we find a contradiction with the fact that $b_n \geq 1$ for all n. To compute the set of solutions, we first find X such that $f(X)/g(X^{D+\delta}) < 1$. Then for each $x \leq X$ and for each y such that $g(y) \leq f(x)$, we check whether the ratio $f(x)/g(y)$ is an integer. $\square$

4.2 Proof of Proposition 1.10. Assume that there exists an infinite sequence $\{(x_n, y_n, b_n)\}$ of solutions in positive integers to the equation $f(x) = bg(y)$ such that for all n, $y_n > x_n^D$. As before, $\lim_{n\to\infty} x_n = \infty$. We have for all n sufficiently large: $b_n = \frac{|f(x_n)|}{|g(y_n)|} \leq \frac{|f(x_n)|}{|g(x_n^D)|}$. Since $\lim_{x\to\infty} f(x)/g(x^D)$ exists, we find that the set $\{b_n, n \in \mathbb{N}\}$ is bounded. There are thus only finitely many curves of the form $f(x) = b_n g(y)$ to consider, and by hypothesis each has a smooth projective model with positive genus. Thus we obtain a contradiction by applying Siegel's Theorem to each such curve to obtain that the union of their integer points is finite. $\square$

Recall that when F is any field and $g(t) \in F[t]$, then the *radical* $\mathrm{rad}(g)$ is defined to be the product of the distinct irreducible factors of g. When $m > 1$ is an integer, the *radical* $\mathrm{rad}(m)$ denotes the product of the distinct primes which divide m. The following proposition assumes that the *abc*-conjecture of Masser and Oesterlé holds ([10, p. 24] or [15, Conjecture 3]).

Proposition 4.3. *Assume that the abc-Conjecture is true. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial without multiple roots. Let $g(y) \in \mathbb{Z}[y]$ be a polynomial with at least one multiple root. Let*

$\delta > 0$. Then the equation $f(x) = bg(y)$ in the variables (x, y, b) has only finitely many solutions (x_0, y_0, b_0) in positive integers with

$$\frac{\log(y_0)}{\log(x_0)} > \frac{1}{\deg(g) - \deg(\mathrm{rad}(g))} + \delta.$$

In particular, if $\frac{\deg(f)}{\deg(g)} - \frac{1}{\deg(g)-\deg(\mathrm{rad}(g))} > 0$, then Conjecture 4.4 holds for f and g with $\epsilon < \frac{\deg(f)}{\deg(g)} - \frac{1}{\deg(g)-\deg(\mathrm{rad}(g))}$. This is the case for instance when $\deg(f) > \deg(g)$, or when $\deg(f) \geq 3$ and $g(y) = y^s$.

Proof. Let $\beta > 0$. Under our assumptions that the *abc*-Conjecture is true and that $f(x)$ does not have repeated roots, we find in [6, Corollary 1 to Theorem 5], that

$$\mathrm{rad}(f(x_0)) := \prod_{\text{primes } p | f(x_0)} p \gg |x_0|^{\deg(f)-1-\beta},$$

where the constant implied in the notation $\gg$ depends on β and f. Consider now a solution (x_0, y_0, b_0) in positive integers of the equation $f(x) = bg(y)$. Then

$$\mathrm{rad}(f(x_0)) = \mathrm{rad}(b_0 g(y_0)) \leq b_0 \mathrm{rad}(g(y_0)) = b_0 \mathrm{rad}(\mathrm{rad}(g)(y_0)) \leq \frac{f(x_0)}{g(y_0)} \mathrm{rad}(g)(y_0) = \frac{f(x_0)}{\frac{g}{\mathrm{rad}(g)}(y_0)}.$$

It follows that there exists a constant c_0 depending on β such that, for all x_0, y_0 sufficiently large,

$$x_0^{\deg(f)-1-\beta} \leq c_0 \frac{x_0^{\deg(f)}}{y_0^{\deg(g)-\deg(\mathrm{rad}(g))}}.$$

Let $d := \deg(g) - \deg(\mathrm{rad}(g))$ and recall that by hypothesis, $d > 0$. Thus, we can write:

$$\frac{\log(y_0)}{\log(x_0)} \leq \frac{1}{d} + \frac{\beta}{d} + \frac{\log(c_0)}{\log(x_0)d}.$$

It follows that if we choose $\delta > 0$, we can find β such that $\frac{\beta}{d} \leq \frac{1}{2}\delta$, and we obtain that for all x_0 large enough (depending of β), we have $\frac{\log(c_0)}{\log(x_0)d} \leq \frac{1}{2}\delta$. If follows that for all solutions (x_0, y_0, b_0) with x_0 and y_0 large enough, we have the desired inequality

$$\frac{\log(y_0)}{\log(x_0)} \leq \frac{1}{d} + \delta.$$

Clearly, for a fixed x_0, the equation $f(x_0) = bg(y)$ has only finitely many integer solutions (y_0, b_0). For a fixed y_0, $\frac{\log(y_0)}{\log(x_0)} \leq \frac{1}{d} + \delta$ as soon as x_0 is large enough. It follows that there can be only finitely many solutions (x_0, y_0, b_0) with $\frac{\log(y_0)}{\log(x_0)} > \frac{1}{d} + \delta$, as desired. □

Proposition 4.3 motivates the following conjecture.

Conjecture 4.4. *Let $f(x) \in \mathbb{Z}[x]$ and $g(y) \in \mathbb{Z}[y]$ be non-constant integer polynomials, and set $D := \deg(f)/\deg(g)$. There exists $\epsilon > 0$, depending on f and g, such that the equation $f(x) = bg(y)$ has only finitely many solutions (x_0, y_0, b_0) in positive integers such that $x_0 > 1$, such that $f(x) - b_0 g(y)$ is geometrically irreducible and the genus of the smooth projective curve associated to the plane curve given by the equation $f(x) = b_0 g(y)$ is positive, and such that $y_0 > x_0^{D-\epsilon}$.*

Example 4.5 Consider a conic with infinitely many integral points, given by an equation $x^2 - dy^2 = -e$, with $d, e \in \mathbb{N}$. Let $f(x) \in \mathbb{Z}[x]$, with $\deg(f) \geq 3$. Assume that $f(x)$ is divisible by $(x^2 + e)$ in $\mathbb{Z}[x]$, and write $f(x) = (x^2 + e)h(x)$ with $\deg(h) > 0$. Then the equation $f(x) = by^2$ has infinitely many solutions (x, y, b) in positive integers with $y > x/\sqrt{d}$. Indeed, for each solution (x_0, y_0) of the equation $x^2 - dy^2 = -e$, we get the equality

$$f(x_0) = dh(x_0)y_0^2.$$

In this example, $\deg(f) \geq 3$ and $\deg(g) = 2$, and we find that for the statement of Conjecture 4.4 to hold for ϵ, we need

$$\frac{\deg(f)}{\deg(g)} - \epsilon > 1.$$

Proposition 4.3 shows that if the *abc*-Conjecture holds, then Conjecture 4.4 is true with $\epsilon < \frac{\deg(f)}{\deg(g)} - 1$. Thus the bound for ϵ provided in 4.3 when $g(y) = y^2$ is sharp.

Remark 4.6 Fix $f(x) \in \mathbb{Z}[x]$ of positive degree. Given $g_0(x) = x^2$, it follows from 4.3 under the abc-conjecture that Conjecture 4.4 holds for the pair $f(x)$ and $g_0(x)$ for $\epsilon < 1/2$. We note here that the same statement does not hold for all quadratic polynomials $g(x)$. For instance, in 5.3, we present an example where $g_1(x) = x(x+1)$ and Conjecture 4.4 can hold for the pair $f(x)$ and $g_1(x)$ only when $\epsilon < 1/6$.

Remark 4.7 Consider the surface $X/\mathbb{Q}$ in the (x, y, b)-affine space given by the equation $f(x) = bg(y)$. We note here that some additional assumption, such as one of the form $\frac{\log(y)}{\log(x)} > \delta$, seems essential in order to obtain a non-trivial finiteness statement for the number of positive integer solutions to $f(x) = bg(y)$. For instance, one might ask, in the spirit of the Lang-Vojta conjecture ([9, F.5.3.6]), whether there exist finitely many irreducible algebraic curves on the surface X such that the complement of the union of these curves in X contains only finitely many integer points. This question is easily shown to have a negative answer when $f(x) = xh(x)$ since in that case, for each integer $e > 0$, the parametric curve C_e given by $(x(y), y, b(y))$ with

$$\begin{aligned} x(y) &:= eg(y) \\ b(y) &:= eh(eg(y)) \end{aligned}$$

lies on the surface X, and contains infinitely many integer points. Note that when in addition $h(0) \neq 0$, then the curves C_e and $C_{e'}$ do not intersect on X when $e \neq e'$.

Conjecture 4.4 predicts the existence of $\epsilon > 0$ such that there exists only finitely many positive solutions (x, y, b) with $\frac{\log(y)}{\log(x)} > \frac{\deg(f)}{\deg(g)} - \epsilon$. Assuming this conjecture, we can then ask a more general question:

What is the largest value of $\epsilon > 0$ *such that there exists only finitely many positive solutions* (x, y, b) *with* $\frac{\log(y)}{\log(x)} > \frac{\deg(f)}{\deg(g)} - \epsilon$ *lying outside of a finite set* S *of curves on the surface* X.

In the example where $f(x) = xh(x)$, the existence of the infinitely many curves C_e each containing infinitely many positive solutions shows that the answer to the more general question can only produce in this case an ϵ with $\epsilon < \frac{\deg(f)}{\deg(g)} - \frac{1}{\deg(g)}$.

Let us now slightly modify Example 4.5 so that we can use an explicit example in [7]. Consider the surface $f(x) = by^2$ in the (x, y, b)-space with $f(x) := (x^2+x+1)h(x)$. Consider the curve C_k in the surface X obtained by intersecting X with the surface given by $x^2+x+1 = ky^2$. It is known [7] that if $k = (q^2 + 3)/4$ for some integer q, then the conic C_k contains infinitely many integral points with positive coordinates. It is easy to check that the conics C_k and $C_{k'}$ have no integer solutions in common when $k \neq k'$.

For each point (x_0, y_0) on C_k having positive coordinates, we can write $f(x_0) = h(x_0)ky_0^2$, so $(x_0, y_0, b_0 := h(x_0)k)$ is an integer point on the surface X such that $\frac{\log(y)}{\log(x)} > 1 - \frac{\log(b_0)}{2\log(x_0)}$. Thus the existence of the infinitely many curves C_k each containing infinitely many positive solutions shows that the answer to the more general question can only produce in this case an ϵ with $\epsilon < \frac{\deg(f)}{\deg(g)} - \frac{1}{\deg(g)-1}$.

5. Upper bounds for ϵ

As mentioned in 1.11, an immediate type of constraint on the possible ϵ's in Conjecture 4.4 comes from the existence of parametric solutions, where a parametric solution of the equation $f(x) = bg(y)$ is a triple $x(t), y(t), b(t)$ of polynomials in $\mathbb{Z}[t]$ with $x(t)$ and $b(t)$ of positive degree such that $f(x(t)) = b(t)g(y(t))$ in $\mathbb{Z}[t]$. We start by providing a proof for Proposition 1.13.

5.1 Proof of Proposition 1.13. The first inequality follows from (1.12). It is clear that the function $\frac{y(t)^{\deg_t(x)}}{x(t)^{\deg_t(y)}}$ has a finite limit when t tends to infinity. Thus, given any $\delta > 0$, there exists a positive integer t_0 such that for all $t_1 > t_0$,

$$\Big|\frac{\log(|y(t_1)|)}{\log(|x(t_1)|)} - \frac{\deg_t(y)}{\deg_t(x)}\Big| < \delta.$$

By hypothesis, the values of the polynomials $x(t), y(t)$, and $b(t)$, are integers when t is an integer. If Conjecture 4.4 holds for ϵ, we see that there exists an integer $t_2 > t_0$ such that

$$\frac{\deg_t(y)}{\deg_t(x)} - \delta < \frac{\log(|y(t_2)|)}{\log(|x(t_2)|)} \leq \frac{\deg(f)}{\deg(g)} - \epsilon.$$

Since this is true for any $\delta > 0$, the result follows. □

Remark 5.2 The hypothesis in Proposition 1.13 that the polynomials $x(t), y(t)$, and $b(t)$, belong to $\mathrm{Int}(\mathbb{Z})$ is needed in its proof. Indeed, consider $f(x) = x(x^2 + x + 1)(x^2 + x + 3)$ and $g(y) = y(y + 1)$. Then $x(t) := t$, $y(t) := \frac{1}{2}(t^2 + t + 1)$, and $b(t) := 4t$, is a solution to $f(x) = bg(y)$ in $\mathbb{Q}[t]$, and $y(t)$ never takes integer values. The existence of this solution does not imply an upperbound on the ϵ for which Conjecture 4.4 holds for f and g. (A computation with Magma shows that there are no other solutions $(x(t), y(t), b(t))$ of $f(x) = bg(y)$ in $\mathbb{Q}[t]$ with $\deg_t(x) = 1$, $\deg_t(y) = 2$ and positive leading coefficients.)

Example 5.3 Consider the equation

$$x(x+1)(x+2) = ty(y+1). \tag{5.4}$$

Let $P := (0,0)$ and $Q := (-2,0)$. One may wonder whether the Mordell-Weil group of the elliptic curve over $\mathbb{Q}(t)$ defined by (5.4) is generated by P and Q. The integral solution $-3P - 3Q = (x(t), y(t))$, with

$$\begin{array}{rcl} x(t) & = & (t^2 - t - 1)(t+1) \\ y(t) & = & (t^2 - t - 1)(t^2 + t - 1), \end{array} \tag{5.5}$$

is of interest since $\frac{\deg_t(y)}{\deg_t(x)} = 4/3$, while $\frac{\deg(f)}{\deg(g)} = 3/2$. If Conjecture 4.4 holds for some $\epsilon > 0$, then Proposition 5.1 implies that $\epsilon \le 3/2 - 4/3 = 1/6$.

The sets of integer solutions of (5.4) when $t = 3$ is considered in [1] (see also [21, Theorem A23]). The problem of finding integer solutions to (5.4) when $t = 1$ was posed by E. Lionnet and solved in ([14], or see [5, page 681]). The full set of integer solutions of (5.4) when $t = 1$ is determined in [13].

It is possible to obtain in some cases an upper bound for $\frac{\deg_t(y)}{\deg_t(x)}$ smaller than $\frac{\deg(f)}{\deg(g)}$ using the ABC-theorem for polynomials (see, e.g., [9, F.3.6] or [10, page 19]). This theorem, also called the Mason-Stothers Theorem ([11] and [24, 1.1]), states that:

Let $a(t)$, $b(t)$, and $c(t)$ be relatively prime polynomials over a field F such that $a + b = c$ and such that not all of them have vanishing derivative. Then

$$\max\{\deg(a), \deg(b), \deg(c)\} \le \deg(\mathrm{rad}(abc)) - 1.$$

When $\mathrm{char}(F) = 0$, we find that if suffices to assume that $a(t)$, $b(t)$, and $c(t)$ are relatively prime polynomials and not all constant.

We use the ABC-theorem and a key lemma of Belyi to obtain the following theorem, whose proof follows ideas found in [6].

Theorem 5.6. *Let $F(x, y) \in \mathbb{Q}[x, y]$ be a homogeneous polynomial. Let $m(t), n(t) \in \mathbb{Q}[t]$ be two coprime polynomials, not both constant. Then*

$$\deg_t(\mathrm{rad}(F(m,n))) - 1 \ge (\deg(\mathrm{rad}(F)) - 2)\max(\deg_t(m), \deg_t(n)).$$

Proof. Belyi's Lemma in the form given in [6, Lemma 1] is used to show the existence of three coprime homogeneous polynomials $a(x, y), b(x, y)$, and $c(x, y)$ in $\mathbb{Z}[x, y]$ of degree $D > 0$ such that $a + b = c$, and such that $\mathrm{rad}(abc)$ has degree $D + 2$ and is divisible by $\mathrm{rad}(F)$. Since $m(t)$ and $n(t)$ are coprime, and so are $a(x, y)$ and $b(x, y)$, we find that $a(m, n)$ and $b(m, n)$ are coprime in $\mathbb{Q}[t]$. Since $m(t)$ and $n(t)$ are not both constant, we find that $a(m, n)$ and $b(m, n)$ cannot be both constant. The ABC-Theorem can then be applied to $a(m, n) + b(m, n) = c(m, n)$ to obtain the following inequality:

$$D\max(\deg_t(m), \deg_t(n)) + 1 \le \deg_t(\mathrm{rad}(a(m,n)b(m,n)c(m,n))).$$

Write $\mathrm{rad}(abc) = \mathrm{rad}(F)(x,y)h(x,y)$ for some $h(x,y) \in \mathbb{Q}[x,y]$. Then

$$\begin{aligned}\deg_t(\mathrm{rad}(a(m,n)b(m,n)c(m,n))) &= \deg_t(\mathrm{rad}(\mathrm{rad}(abc)(m,n)))\\ &= \deg_t(\mathrm{rad}(\mathrm{rad}(F)(m,n)h(m,n))))\\ &\leq \deg_t(\mathrm{rad}(F(m,n))) + \deg(h)\max(\deg_t(m), \deg_t(n))\end{aligned}$$

Putting these inequalities together, along with the expression $\deg(h) = (D+2) - \deg(\mathrm{rad}(F))$, concludes the proof. ☐

Corollary 5.7. *Let $f(x) \in \mathbb{Q}[x]$. Let $m(t) \in \mathbb{Q}[t]$ be of positive degree. Then*

$$\deg_t(\mathrm{rad}(f(m))) \geq 1 + \deg_t(m)(\deg(\mathrm{rad}(f)) - 1).$$

Proof. Apply Theorem 5.6 to $F(x,y) := y^{\deg(f)+1} f(x/y)$, and to the polynomials $m(t)$ and $n(t) = 1$. ☐

Note that the inequality in the corollary becomes an equality when 0 is a simple root of f and $m(t) = t^s$.

Theorem 5.8. *Let $f(x) \in \mathbb{Q}[x]$ and $g(y) \in \mathbb{Q}[y]$. Let $(x_0(t), y_0(t), b_0(t))$ be a solution of the equation $f(x) = bg(y)$ with $x_0(t), y_0(t), b_0(t) \in \mathbb{C}[t]$ and $x_0(t)$ not constant. Then*

$$(\deg(g) - \deg(\mathrm{rad}(g)))\deg_t(y_0) \leq (\deg(f) - \deg(\mathrm{rad}(f)) + 1)\deg_t(x_0) - 1, \tag{5.9}$$

and

$$(\deg(\mathrm{rad}(g)) - 1)\deg_t(y_0) \leq \deg(\mathrm{rad}(f))\deg_t(x_0) - 1. \tag{5.10}$$

In particular,

(a) *If g has at least one multiple root, then*

$$\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(f) - \deg(\mathrm{rad}(f)) + 1}{\deg(g) - \deg(\mathrm{rad}(g))}.$$

If in addition $\deg(f) > \deg(g)$ and f has no multiple roots, then $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{1}{\deg(g) - \deg(\mathrm{rad}(g))} < \frac{\deg(f)}{\deg(g)}$.

(b) *If $\deg(\mathrm{rad}(g)) > 1$, then*

$$\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(\mathrm{rad}(f))}{\deg(\mathrm{rad}(g)) - 1}.$$

If in addition f has at least one multiple root, g has no multiple root, and $\deg(f) < \deg(g)$, then $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(f)-1}{\deg(g)-1} < \frac{\deg(f)}{\deg(g)}$. Similarly, if in addition $f(x) = x^r$ and $\deg(f) > \deg(g)$, then $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{1}{\deg(\mathrm{rad}(g))-1} < \frac{\deg(f)}{\deg(g)}$.

Proof. We apply Corollary 5.7 to $f(x)$ and $x_0(t)$, to obtain the first inequality below:

$$\begin{aligned}
1+\deg_t(x_0)(\deg(\mathrm{rad}(f))-1) &\leq \deg_t(\mathrm{rad}(f(x_0)))\\
&= \deg_t(\mathrm{rad}(b_0 g(y_0))\\
&\leq \deg_t(b_0)+\deg_t(\mathrm{rad}(g(y_0))\\
&= \deg_t(b_0)+\deg_t(\mathrm{rad}(\mathrm{rad}(g)(y_0))\\
&\leq \deg_t(b_0)+\deg(\mathrm{rad}(g))\deg_t(y_0)\\
&= \deg(f)\deg_t(x_0)-\deg(g)\deg_t(y_0)+\deg(\mathrm{rad}(g))\deg_t(y_0),
\end{aligned}$$

and (5.9) follows. To prove (5.10), we apply Corollary 5.7 to $g(y)$ and $y_0(t)$ to obtain the first inequality below:

$$\begin{aligned}
(\deg(\mathrm{rad}(g))-1)\deg_t(y_0) &\leq \deg_t(\mathrm{rad}(g(y_0)))-1\\
&\leq \deg_t(\mathrm{rad}(b_0 g(y_0))-1\\
&= \deg_t(\mathrm{rad}(f(x_0))-1\\
&\leq \deg(\mathrm{rad}(f))\deg_t(x_0)-1.
\end{aligned}$$

□

We can slightly improve Theorem 5.8, in the special case where $x^r = bg(y)$ and $r \leq \deg(g)$, using the main result of [12].

Proposition 5.11. *Let F be a field of characteristic 0. Let $f(x) = x^r$, $r > 1$, and $g(y) \in F[y]$. Assume that the equation $x^r = g(y)$ defines a smooth projective geometrically connected curve over F of positive genus. Then there exists $\epsilon > 0$ depending on $\deg(f)$ and $\deg(g)$ only such that, given any solution $(x_0(t), y_0(t), b_0(t))$ to $x^r = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ not constant, then $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(f)}{\deg(g)} - \epsilon$.*

Proof. Given a solution $(x_0(t), y_0(t), b_0(t))$, we let $\overline{g}(y) := b_0(t)g(y) \in F[t][y]$. We apply to the equation $x^r = \overline{g}(y)$ the main result of [12, Theorem on page 168] to obtain the following inequality:

$$\deg_t(y_0) \leq 78\deg_t(b_0) + 6 \leq 79\deg_t(b_0). \tag{5.12}$$

In the notation of [12, page 168] we apply the Theorem to the case where $K = F(t)$ with $g_K = 0$, and where S consists only in the place corresponding to the point at infinity, with local ring $F[1/t]_{(1/t)}$, so that $\mathcal{O}_S = F[t]$. The proposition then follows from the fact that (c) implies (a) in Remark 5.13. □

Remark 5.13 Let F be any field. Let $f(x)$ and $g(x)$ be non-constant polynomials in $F[x]$. Then the following statements are equivalent:

(a) There exists $\epsilon > 0$ such that $\frac{\deg_t(y_0)}{\deg_t(x_0)} < \frac{\deg(f)}{\deg(g)} - \epsilon$ for any solution $(x_0(t), y_0(t), b_0(t))$ to the equation $f(x) = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ non-constant.

(b) There exists $C_1 > 0$ such that $\frac{\deg_t(x_0)}{\deg_t(b_0)} < C_1$ for any solution $(x_0(t), y_0(t), b_0(t))$ to the equation $f(x) = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ non-constant.

(c) There exists $C_2 > 0$ such that $\frac{\deg_t(y_0)}{\deg_t(b_0)} < C_2$ for any solution $(x_0(t), y_0(t), b_0(t))$ to the equation $f(x) = bg(y)$ in polynomials in $F[t]$ with $x_0(t)$ and $b_0(t)$ non-constant.

The equivalences follow from the equality

$$\deg(f)\deg_t(x_0) = \deg_t(b_0) + \deg(g)\deg_t(y_0).$$

The equivalence of (b) and (c) is immediate. To show that (c) implies (a), we note that

$$\frac{\deg_t(y_0)}{\deg_t(x_0)} = \frac{\deg(f)}{\deg(g)} - \frac{\deg_t(b_0)}{\deg(g)\deg_t(x_0)} < \frac{\deg(f)}{\deg(g)} - \frac{\deg_t(y_0)}{C_2\deg(g)\deg_t(x_0)},$$

and we can take

$$\epsilon = \frac{\deg(f)}{(C_2\deg(g)+1)\deg(g)}.$$

Assuming (a), we find that $\frac{\deg_t(x_0)}{\deg_t(b_0)} < \frac{1}{\epsilon\deg(g)}$.

6. Distribution of the integer solutions

We report in this section on some data on the solutions (x, y, b, c) in positive integers with $\gcd(b, c) = 1$ to the equation

$$cx(x+1)(x+2)(x+3) = by(y+1). \tag{6.1}$$

In 6.6, we consider all positive integer solutions (x, y, b, c) to Equation 6.1 that we found, about 369,000 of them, with $c \in [1, 300]$, $x \in [1, 10^9]$, $\gcd(b, c) = 1$, $b \neq 4c$ and $\frac{\log(y)}{\log(x)} \geq 1.33333$. We remove from this set of solutions all solutions which can be explained 'geometrically', that is, solutions which we found to belong to a parametric family. (Finding parametric families of solutions is not immediate, and we explain below how some of these families were found.) We introduce then the following counting function for this set of 'not-geometrically explained' solutions:

$$N(B) \;:=\; \text{Number of solutions } (x,y,b,c) \text{ with } \gcd(b,c)=1, b\neq 4c, \frac{\log(y)}{\log(x)} \in [4/3, B], x\in[1,10^9], \text{ and } c\in[1,300].$$

The data shows in 6.7 a surprisingly good fit between $N(B)$ and a function of the form $\alpha - \beta e^{-\gamma B}$, where α, β, γ are positive constants.

6.2 For each positive integer c, let $f_c(x) := cx(x+1)(x+2)(x+3)$. If Conjecture 4.4 holds, then there exists ϵ_c such that the equation $f_c(x) = by(y+1)$ has only finitely many solutions (x_0, y_0, b_0) in positive integers with $b_0 \neq 4c$ and $y_0 \geq x_0^{D-\epsilon_c}$.

A computer search found the complete list $L = L_{7/4}$ of all 12730 solutions (x, y, b, c) to Equation (6.1) in positive integers with $c \in [1, 300]$, $b \neq 4c$, $x \leq 10^9$, $\gcd(b, c) = 1$, and $\frac{\log(y)}{\log(x)} > 7/4$. The parametric solution

$$x(c) = 144c + 2, \quad y(c) = 24c(144c+5), \quad b(c) = 36c+1, \tag{6.3}$$

has $\frac{\log(y)}{\log(x)} > 7/4$ when $c \geq 9$. For 156 values of c when $c \in [9, 300]$, the solution (x, y, b, c) with the largest x-coordinate for the given c among the solutions in L is the solution given in the parametric family (6.3) above. We found only 232 solutions in L with x-coordinate larger than the x-coordinate of the corresponding solution in the parametric family (6.3). Our data supports the statement that Conjecture 4.4 holds for any f_c with $\epsilon_c < 1/4$.

The equation (1.9) has some parametric solutions of interest with $\frac{\deg_t(y)}{\deg_t(x)} = 3/2$:

$x(t)$	$y(t)$	$b(t)$
$(t+1)(2t-3)$	$t(2t^2-t-2)$	$(2t+1)(2t-3)$
$(t-1)(2t+3)$	$(t-1)(t+1)(2t+1)$	$(2t-1)(2t+3)$

Except for when $t = 1, 2$, these parameterizations produce solutions (x, y, b) with $\frac{\log(y)}{\log(x)} < 3/2$, and $\lim_{t\to\infty} \frac{\log(y)}{\log(x)} = 3/2$. If Conjecture 4.4 holds for $f_c(x) = by(y+1)$ for some $\epsilon_c > 0$, then Proposition 1.13 implies that $\epsilon_c < 1/2$.

We found a total of 36 solutions (x, y, b, c) in L with $x > 10^6$. We only found two values of c with solutions with $10^8 < x < 10^9$, when $c = 39$ and when $c = 281$:

x	y	b	c	$\log(y)/\log(x)$
169036273	2130028657415099	7018	39	1.86
554344548	2537086889346525	4122444	281	1.76

Remark 6.4 It would be of interest to determine the supremum of the set $\mathcal{D}$ of all fractions $\frac{\deg_t(y)}{\deg_t(x)}$ where $(x(t), y(t), b(t))$ is a solution of the equation $x(x+1)(x+2)(x+3) = by(y+1)$ with polynomials in $\mathbb{Q}[t]$ and $\deg_t(x), \deg_t(b) > 0$. As noted in (1.12), $\mathcal{D}$ is bounded by $D = 2$. We can show that $\{\frac{1}{2}, \frac{2}{3}, \frac{n+2}{n+3}, n \in \mathbb{N}, 1, \frac{4}{3}, \frac{3}{2}\}$ is contained in $\mathcal{D}$. We only provide examples below to show that 1 and 4/3 belong to $\mathcal{D}$.

First, note that using the solution $(x(t), y(t), t)$ in (5.5), we obtain two solutions of Equation (1.9) with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{4}{3}$, namely

$$(x(t), y(t), t(x(t)+3)) \quad \text{and} \quad (x(t)-1, y(t), t(x(t)-1)). \tag{6.5}$$

A much less obvious solution $(x(t), y(t), b(t))$ to Equation (1.9) with polynomials in $\mathbb{Q}[t]$ and with $\frac{\deg_t(y)}{\deg_t(x)} = \frac{4}{3}$ is as follows:

$$\begin{aligned} x &= \tfrac{1}{54}(t-6)(t^2+8t+21), \\ y &= \tfrac{1}{216}(t-6)(t+3)(t^2+5t-12), \\ b &= \tfrac{4}{729}(t^2-4t-3)(t^2+8t+21). \end{aligned}$$

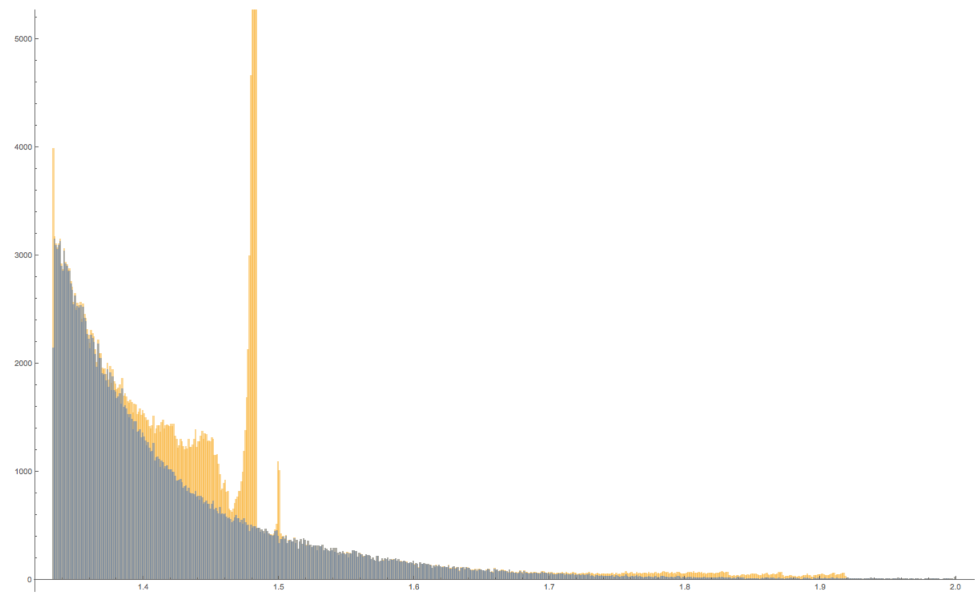
This solution was obtained by carefully considering the solutions (x, y, b, c) in positive integers to $cx(x+1)(x+2)(x+3) = by(y+1)$ with $\frac{\log(y)}{\log(x)}$ around 4/3. Using the changes of variables $t = 36s-3$ or $t = 108s+6$, we obtain two solutions $(x(s), y(s), b(s))$ in integer polynomials to the equation $27x(x+1)(x+2)(x+3) = by(y+1)$. Composing further the change $t = 36s - 3$ with the change $s = 27r + 5$ produces the following solution $(x(r), y(r), b(r))$ in integer polynomials to the original Equation (1.8) $x(x+1)(x+2)(x+3) = by(y+1)$:

$$\begin{aligned} x &= (108r+19)(157464r^2+58644r+5461), \\ y &= 27(27r+5)(108r+19)(52488r^2+19386r+1789), \\ b &= 16(1944r^2+700r+63)(157464r^2+58644r+5461). \end{aligned}$$

It is easy to write down solutions to Equation (1.9) with $\deg_t(y) = \deg_t(x) = 1$, such as $y(t) = t$ and $x(t) \in \{t-2, t-1, t, 2t-1, 2t, 3t\}$. We note that there are also at least eight solutions with $\deg_t(y) = \deg_t(x) = 2$ which are not obtained by composition from solutions with $\deg_t(y) = \deg_t(x) = 1$; one such solution has $x(t) = \frac{1}{2}t(t+5)$ and $y(t) = \frac{1}{2}t(t+3)$.

6.6 Data when $\frac{\log(y)}{\log(x)} \geq 1.33333$.

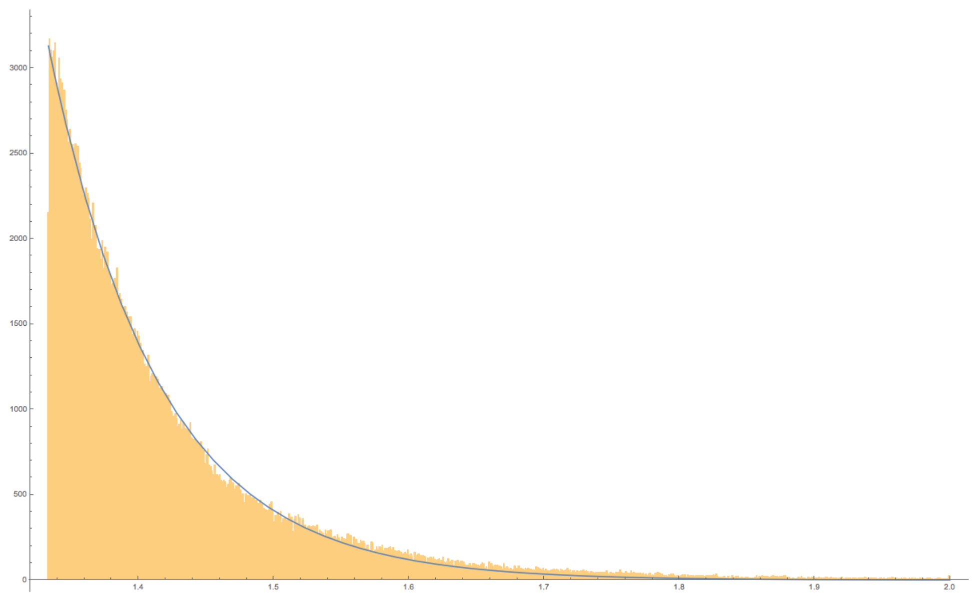
A second computer search found the complete list L of all solutions (x, y, b, c) to Equation (6.1) in positive integers with $c \in [1, 300]$, $b \neq 4c$, $x \leq 10^9$, $\gcd(b, c) = 1$, and $\frac{\log(y)}{\log(x)} \geq 1.33333$. We present a histogram below of this data produced using Mathematica [26]. We partitioned the interval $[1.33333, 2]$ in intervals I_d of length 0.001 and for each d counted the number of solutions found with $\frac{\log(y)}{\log(x)} \in I_d := [d, d+0.001)$. Thus in the histogram below, the horizontal axis pertains to the quantity $\frac{\log(y)}{\log(x)}$ and the vertical axis corresponds to the number of solutions found.



The parametric solutions to Equation (6.1) with $c = 1$ found in 6.2 explain the very tall peak around $\frac{\log(y)}{\log(x)} = 1.47$. Note that the histogram is truncated at height around 5000 and that the very tall peak extends to about height 30,000. The histogram is also truncated on the x-axis to $\frac{\log(y)}{\log(x)} \leq 2$.

The histogram also shows a smaller sharp peak in the number of solutions around $\frac{\log(y)}{\log(x)} = 1.5$. This peak is explained geometrically by the existence of the six parametric solutions in 6.10 below. The smaller sharp peak around around $\frac{\log(y)}{\log(x)} = 4/3$ is explained by the parametric solutions in (6.5). The 'bulge' in the number of solutions above the interval $[1.40, 1.46]$ is also explained by the existence of many other parametric solutions, as we explain in 6.11. The very thin bulge above the interval $[1.7, 2]$ is explained by the parametric solutions described in (6.9).

6.7 Remove now from the set L all possible positive integer solutions (x, y, b, c) which belong to one of the parametric solutions generated by the solutions discussed in this section. We present a histogram below of this data. We partitioned the interval $[1.33333, 2]$ in intervals I_d of length 0.001 and for each d counted the number of solutions found with $\log(y)/\log(x) \in I_d := [d, d + 0.001)$. The blue graph on this histogram is the graph of the function $h(B) := \exp(\alpha(B-2)+\beta)$, with $\alpha = -12.2368$ and $\beta = -0.1092$ obtained using a Mathematica command to produce the exponential function that fits best the data on this histogram when $t \in [1.5002, 2]$.



6.8 We end this section by describing the parametric solutions that we found that explain the bulge in the first histogram. Given any parametric solution $(x(c), y(c), b(c), c)$, we can use the involutions $x \mapsto -x-3$ and $y \mapsto -y-1$ if necessary to obtain a new solution where the leading coefficients of both $x(c)$ and $y(c)$ are positive. Consider then the set S of all parametric solutions $(x(c), y(c), b(c))$ with degrees $(1, 2, 1)$, and $x(c)$ and $y(c)$ having positive leading coefficients (such as (6.3)). Then the involution

$$I : (x(c), y(c), b(c)) \longmapsto (-x(-c) - 3, y(-c), -b(-c))$$

preserves this set of solutions. Let $x(c) := a_1 c + a_0$, $y(c) := c_2 c^2 + c_1 c + c_0$, and $b(c) := b_1 c + b_0$ be polynomials in $\mathbb{Q}[c]$ such that $(x(c), y(c), b(c), c)$ is a solution with $a_1, c_2 > 0$ and $b_0 \neq 0$. Since $\deg_c(x) = 1$, we can find a new solution in $\mathbb{Q}[c]$ such that $x(c)$ is monic. It turns out that the set of all such solutions can be explicitly computed, and is given by the following four solutions and their images under the involution I. In the tables of solutions below, we let $z(300) := \frac{\log(y(300))}{\log(x(300))}$.

(6.9)

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$c+2$	$\frac{c(c+5)}{6}$	$36c+144$	1.68680
$c+1$	$\frac{c(c+4)}{3}$	$9c+18$	1.80866
$c+1$	$\frac{c(c+3)}{2}$	$4c+16$	1.87912
$c-1$	c^2-1	$c+2$	2.00117

Using Magma [3], one computes that the ideal in $A := \mathbb{Q}[a_0, b_0, b_1, c_0, c_1, c_2]$ generated by the coefficients of the polynomial $cx(x+1)(x+2)(x+3) - by(y+1)$ in $A[c]$ has exactly 18 distinct minimal prime ideals. The 16 prime ideals which do not contain b_0c_2 are maximal. Exactly half of these determine a coefficient c_2 which is non-negative.

To obtain solutions to Equation 6.1 in positive integers, we evaluate the above solutions at integer values c_0 such that $y(c_0) \in \mathbb{Z}$. For instance, using the first solution above, for each appropriate values of c_0 we obtain the integer solution (x, y, b, c) given as

$$x = c_0 + 2, y = \frac{c_0(c_0+5)}{6}, b = \frac{36c_0 + 144}{\gcd(c_0, 144)}, c = \frac{c_0}{\gcd(c_0, 144)}.$$

From this, we can write down explicit integer parametrizations of solutions. For instance, by setting $c_0 = 144d$ in the above expression we obtain (6.3), and by setting $c_0 = 6d+1$, we obtain:

$$x = 6d+3, y = (6d+1)(d+1), b = 36(6d+5), c = 6d+1.$$

When $c = 90, 96, 144, 150, 162$, we found only eight solutions in positive integers to (6.1) with $x < 10^9$ and $\log(y)/\log(x) > 7/4$, and all solutions that we found are obtained from the above parametric solutions (6.9) and their images under I, specialized to the case where $c_0 = b(0)d$.

6.10 The following are solutions $(x(c), y(c), b(c))$ to Equation (6.1), with degrees $(2, 3, 3)$.

$x(c)$	$y(c)$	$b(c)$	$z(300)$
c^2+c-3	$(c+1)(c^2+c-1)$	$(c-1)(c^2+c-3)$	1.50015
c^2+c-2	$(c+1)(c^2+c-1)$	$(c-1)(c^2+c+1)$	1.50015
$9c^2+3c-3$	$(3c+1)(9c^2+3c-1)$	$(3c-1)(3c^2+c-1)$	1.50004
c^2-c-3	$c^2(c-2)$	$(c+1)(c^2-c-3)$	1.49986
c^2-c-2	$c^2(c-2)$	$(c+1)(c^2-c+1)$	1.49986
$9c^2-3c-3$	$9c^2(3c-2)$	$(3c+1)(3c^2-c-1)$	1.49996

Consider the set of polynomial solutions $(x(c), y(c), b(c))$ to Equation (6.1) with degrees $(2, 3, 3)$, and $x(c)$ and $y(c)$ having positive leading coefficients. The involution

$$I : (x(c), y(c), b(c)) \longmapsto (x(-c), -y(-c) - 1, -b(-c))$$

preserves this set of solutions, and the last three solutions above are obtained from the first three by this involution.

We note that the first solution above is such that $b(0) = 3$. It follows that for integers c_0 divisible by 3, $\gcd(b(c_0), c_0) \neq 1$. On the other hand, we may consider the new solution $(x(3c), y(3c), b(3c)/3)$, still with integer polynomials, which has the property that $\gcd(b(c_0)/3, c_0) = 1$ for all integer values of c_0. The third solution above is obtained from the first solution by this process.

6.11 The first two solutions in 6.10 are of the form $(x-1, y, b(x-1))$ and $(x, y, b(x+3))$ where $(x, y, b) := ((c-2)(c+1), c^2(c-2), c+1)$ is a solution to the equation $cx(x+1)(x+2) = by(y+1)$. We found three additional solutions to Equation (6.1) coming from this modified equation, listed below. Obviously, their images under the involution I are also solutions.

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$2c^2+5c+1$	$c(c+2)(2c+3)$	$(2c+1)(x+3)$	1.47131
c^2+4c+2	$\frac{c(c+2)(c+3)}{2}$	$4(c+2)(x+3)$	1.43902
c^2+4c+1	$\frac{c(c+2)(c+3)}{2}$	$4(c+2)x$	1.43902

We list below solutions to Equation (6.1) obtained from solutions to the equation $cx(x+1)(x+3) = by(y+1)$.

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$3c^2+7c+1$	$c(c+2)(3c+4)$	$(3c+1)(x+2)$	1.45606
c^2-4	$\frac{c(c-2)(c+1)}{2}$	$4(c+2)(x+1)$	1.43895
c^2+5c+3	$\frac{c(c+2)(c+4)}{3}$	$9(c+3)(x+2)$	1.40340
$3c^2+8c+2$	$\frac{c(c+2)(3c+5)}{2}$	$4(3c+2)(x+1)$	1.40064
$(c+3)(2c+1)$	$\frac{c(c+2)(2c+5)}{3}$	$9(2c+3)(x+1)$	1.38048

It is an easy matter for Magma to compute the parametric solutions (x, y, b) of degree $(2, 3, 1)$ to $cx(x+1)(x+2) = by(y+1)$ and to $cx(x+1)(x+3) = by(y+1)$, and each such solution produces solutions to Equation (6.1). We list below a set of integral parametric solutions to (6.1) which are not of that form. (We leave it to the reader to produce further such solutions using the involution I, or a substitution $c \mapsto \alpha c$ when appropriate.)

$x(c)$	$y(c)$	$b(c)$	$z(300)$
$4(24c^2+13c+1)$	$c(8c+3)(24c+7)$	$8(4c+1)(6c+1)(12c+5)$	1.40049
$(4c+3)(8c+1)$	$c(8c+3)(8c+5)$	$8(2c+1)(4c+1)(4c+3)$	1.43004
$6c^2-11c+3$	$(c-1)(2c-1)(3c-1)$	$(2c-3)(3c-4)(6c-5)$	1.43233
$2(12c^2+11c+1)$	$c(4c+3)(12c+5)$	$4(2c+1)(3c+2)(6c+1)$	1.43854

A priori, finding all polynomial solutions (x, y, b) of degrees $(2, 3, 3)$ is a problem with 7 variables (the coefficients of $x(c)$ and $y(c)$), and 6 relations (since the remainder of the division of $cf(x(c))$ by $y(c)(y(c)+1)$ has degree at most 5), and such a problem seem computationally hard at this time. As we now explain, to find the four solutions listed above, we considered instead a different problem where the number of variables and the degrees of the relations can be considerably decreased.

We may assume that c divides $y(y + 1)$, otherwise we are reduced to an easier problem. In fact, we may assume that c divides y by using the involution $y \mapsto -y - 1$ if necessary. Given a solution (x, y, b) to $cx(x + 1)(x + 2)(x + 3) = by(y + 1)$ with $\deg_c(x) = 2$, we claim that one of the polynomials x, $x + 1$, $x + 2$, and $x + 3$, must be irreducible in $\mathbb{Q}[c]$. Indeed, otherwise we find that $\Delta := a_1^2 - 4a_2a_0$ is a square in $\mathbb{Z}$, and so are $\Delta - 4a_2$, $\Delta - 8a_2$, and $\Delta - 12a_2$. Since we are looking for a solution with $a_2 \neq 0$, it follows that we would then have a 4-term arithmetic progression in squares, and this was proved not to exist by Fermat [23]. Hence, we may assume that $x + i$ is irreducible, and that $x + i$ divides y or $y + 1$ (if this were not the case, $x + i$ would divide b, and we would be reduced to an easier case). Thus, either $y(c) = tc(x(c) + i)$, and there are only 4 variables in total, a_0, a_1, a_2 and t, or $y(c) + 1 = (x(c) + i)(tc + s)$, with $1 = (x(0) + i)s$. This determines s, and leaves only 4 variables in total again, a_0, a_1, a_2 and t.

References

[1] E. Avanesov, *Solution of a problem on figurate numbers* (Russian), Acta Arith. **12** 1966/1967, 409–420. 16

[2] Yuri Bilu and Robert Tichy, *The Diophantine equation $f(x) = g(y)$*, Acta Arith. **95** (2000), no. 3, 261–288. 2

[3] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265. `http://magma.maths.usyd.edu.au/magma/` 7, 8, 23

[4] D. Boyd and H. Kisilevsky, *The Diophantine equation $u(u+1)(u+2)(u+3) = v(v+1)(v+2)$*, Pacific J. Math. **40** (1972), 23–32. 1, 7

[5] L. Dickson, *History of the theory of numbers*, Vol. II: Diophantine analysis. Reprint of the 1920 original, Chelsea Publishing Co., New York 1966. 16

[6] A. Granville, *ABC allows us to count squarefrees*, Internat. Math. Res. Notices 1998, no. 19, 991–1009. 13, 16

[7] Emil Grosswald, *On a classical Diophantine equation*, Boll. Un. Mat. Ital. (4) **1** (1968), 382-392. 15

[8] P. Heichelheim, *Some remarks on Stolt's theorems for Pellian equations*, Ark. Mat. **12** (1974), 167–171. 9

[9] Marc Hindry and Joseph Silverman, *Diophantine geometry. An introduction*, Graduate Texts in Mathematics **201**, Springer-Verlag, New York, 2000. 3, 14, 16

[10] Serge Lang, *Math talks for undergraduates*, Springer-Verlag, New York, 1999. 12, 16

[11] R. C. Mason, *Diophantine equations over function fields*, London Mathematical Society Lecture Note Series, 96. Cambridge University Press, Cambridge, 1984. 16

[12] R. C. Mason and B. Brindza, *LeVeque's superelliptic equation over function fields*, Acta Arith. **47** (1986), no. 2, 167–173. 18

[13] L. J. Mordell, *On the integer solutions of $y(y + 1) = x(x + 1)(x + 2)$*, Pacific J. Math. **13** Number 4 (1963), 1347–1351. 16

[14] Claude S. Moret-Blanc, in *Solutions de questions proposées dans les Nouvelles annales*, Nouvelles annales de mathémathiques **20** Serie 2 (1881), 425–432. 16

[15] Joseph Oesterlé, *Nouvelles approches du théorème de Fermat*, Séminaire Bourbaki **30** (1987-1988), Exp. No. 694. 12
[16] OEIS Foundation Inc. (2011). The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`. 9, 10
[17] W. A. Stein et al., *Sage Mathematics Software (Version 5.11)*, The Sage Development Team, 2013, `http://www.sagemath.org` 7
[18] Pierre Samuel, *Compléments à un article de Hans Grauert sur la conjecture de Mordell*, Inst. Hautes Études Sci. Publ. Math. **29** (1966), 55–62. 4
[19] N. Saradha and T. N. Shorey, *On the ratio of two blocks of consecutive integers*, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), no. 2, 107–132. 1
[20] B. Stolt, *On the Diophantine equation $u^2 - Dv^2 = \pm 4N$. III.* Ark. Mat. **3** (1955), 117–132. 9
[21] R. Stroeker and B. de Weger, *Elliptic Binomial Diophantine Equations*, Math. Comp. **68** (1999), no. 227, 1257–1281. 1, 2, 7, 16
[22] S. Tengely, *On the Diophantine equation $F(x) = G(y)$*, Acta Arith. **110** (2003), no. 2, 185–200. 3
[23] Alf Van Der Poorten, *Fermat's Four Squares Theorem*, `http://arxiv.org/abs/0712.3850`, 2007. 25
[24] W. W. Stothers, *Polynomial identities and Hauptmoduln*, Quart. J. Math. Oxford Ser. (2) **32** (1981), no. 127, 349–370. 16
[25] José Felipe Voloch, *Diophantine geometry in characteristic p: a survey*, Arithmetic geometry (Cortona, 1994), 260–278, Sympos. Math., XXXVII, Cambridge Univ. Press, Cambridge, 1997. 4
[26] Wolfram Research, Inc., *Mathematica*, Version 10.3, Champaign, IL (2015). 21
[27] Z. Xiang, *Nonexistence of nontrivial tight 8-designs*, J. Alg. Comb. **47** (2018), 301–318. 1

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