# Classification of spherical 2-distance $\{4, 2, 1\}$-designs

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**Abstract**

Given a set $T$ of positive integers, spherical $T$-design is a generalization of the concept of the classical spherical $t$-design. In this paper we classify finite subsets that are spherical 2-distance sets and spherical $\{4, 2, 1\}$-designs at the same time. There are two such families: tight spherical 4-designs and halves of tight spherical 5-designs. The key step of the proof is deriving a certain complicated Diophantine equation and then finding its integer solutions under some conditions with the assistant of computer calculations.

# 1 Introduction

The study of *spherical designs* can be tracked from Delsarte-Goethals-Seidel [DGS77]. Bannai-Okuda-Tagami [BOT15] started to study *spherical designs of harmonic index* $T$, or *spherical* $T$*-designs*, where $T$ is a set of positive integers. They are generalizations of spherical $t$-designs. Their definition will be given in § 2. Barg et al. studied in [BGOY15] a finite subset on the unit sphere which is a 2-distance set and a tight frame (i.e. spherical $\{2\}$-design) at the same time. It is proved in [BGOY15, Theorem 1] that such a finite subset is either a spherical embedding of a strongly regular graph (or SRG for short), or a shifted spherical embedding of an SRG, or an equiangular tight frame.

The main result of this paper is a classification of finite sets $X$ which are spherical 2-distance sets and spherical $\{4, 2, 1\}$-designs at the same time. It is known in [DGS77] that since $X$ is a spherical 2-distance $\{2, 1\}$-design, $X$ is a spherical embedding of an SRG. Moreover, $X$ being a spherical $\{4\}$-design allows us to classify all the possible SRGs involved and give the following classification result.

**Theorem 1.1.** *Let $X \subset S^{n-1}$ be a finite subset where $n \geq 2$. Suppose that $X$ is a spherical 2-distance $\{4, 2, 1\}$-design. Then, one of the following holds:*

(i) *$X$ is a tight spherical 4-design on $S^{n-1}$;*

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(ii) *the disjoint union* $X \cup -X$ *is a tight spherical* 5*-design on* $S^{n-1}$*.*

In case (ii), $X$ can be regarded as a half of a tight spherical 5-design. However, it is unknown if there exists a good way to choose a half of a tight spherical 5-design to get $X$. More precisely, it might be a difficult problem to choose a point from each antipodal pair of points in a tight spherical 5-design to get a subset that is a spherical 2-distance $\{4, 2, 1\}$-design. Partly motivated by this result, we studied in [BZZZZ18] when a half of an antipodal spherical $t$-design becomes a spherical 1-design.

In Theorem 1.1, tight spherical 4-designs and tight spherical 5-designs appear. The classification of tight spherical designs has been studied for several decades [DGS77; BD79; BD80; BMV05; NV13], but tight spherical 4-designs and tight spherical 5-designs are not classified yet.

The proof of Theorem 1.1 is given in § 4, and the strategy of the proof is the following. By the discussion above, it suffices to classify subset that is a spherical embedding of an SRG and a spherical $\{4\}$-design at the same time. We first analyze the behavior of spherical embeddings of SRGs in § 3, and reduce the problem down to solving a fairly complicated Diophantine equation in three variables assuming some specific conditions from combinatorial properties of SRGs. Then, we solve this Diophantine equation with additional conditions in Proposition 4.1 and Theorem 5.1, and determine all the feasible parameters of SRGs involved. Since the proof of Theorem 5.1 is too technical, it is given in § 5 and is written in a way that it does not rely on any results in previous sections.

The determination of all these feasible parameters is the most crucial part of the proof of the main result. The effective use of computer calculations plays an important role, and we think there might exist possible wide applications of this approach. We used Mathematica extensively in our calculations and we will explain what we use and how we use the computer in § 5.1. (So the correctness of our present proof depends on whether the calculations of Mathematica is completely reliable or not.) There may be some discussions whether we can avoid this much use of computer. In principle, it is possible, but would be very complicated. Although we could determine all the feasible parameters, it should be emphasized that it still remains as an interesting open problem whether there exist such SRGs for each remaining feasible parameter.

A related interesting question is to classify a set $X$ which is spherical 3-distance 5-design. Based on numerical experiments, we give the following conjecture.

**Conjecture 1.2.** Let $X \subset S^{n-1}$ be a finite subset where $n \geq 2$. If $X$ is a spherical 3-distance 5-design, then one of the following holds:

(i) $n = 2$ and $X$ is the vertices of a regular hexagon or a regular heptagon;

(ii) $X$ is a tight spherical 5-design on $S^{n-1}$;

(iii) $X$ is a section of a tight spherical 7-design on $S^{n-1}$.

In case (iii), there are four distinct inner product values $\{-1, 0, \pm\alpha\}$ between distinct points in the tight spherical 7-design. A section means the collection of points that have the same inner product value $\alpha$ with a fixed point in the design.

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## 2 Spherical design of harmonic index $T$

Let $S^{n-1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ be the real unit sphere in the $n$-dimensional Euclidean space $\mathbb{R}^n$. A finite subset $X$ of the unit sphere is an *$s$-distance set* if there are exactly $s$ distinct distances (equivalently, inner products) between two distinct points in $X$, namely the set

$$A(X) := \{\langle x, y\rangle \mid x, y \in X, x \neq y\},$$

has cardinality $s$, where $\langle\ ,\ \rangle$ is the usual Euclidean inner product.

Let $T$ be a set of some positive integers. A finite subset $X \subset S^{n-1}$ is called a *spherical design of harmonic index $T$*, or simply *spherical $T$-design*, on $S^{n-1}$ if

$$\sum_{x \in X} f(x) = 0$$

holds for every homogeneous harmonic polynomial $f$ of degree $t$ with $t \in T$. It is well known that $X$ is a spherical $T$-design on $S^{n-1}$ if and only if

$$\sum_{x,y \in X} Q_{n,t}(\langle x, y\rangle) = 0 \quad \text{for every } t \in T, \tag{2.1}$$

where $Q_{n,t}(\xi)$ is the Gegenbauer polynomial of degree $t$ in one variable $\xi$. In this paper, $Q_{n,t}(\xi)$ is normalized so that $Q_{n,t}(1) = \binom{n+t-1}{n-1} - \binom{n+t-3}{n-1}$. This value equals the dimension of the vector space of all homogeneous harmonic polynomials of degree $t$ in $n$ variables.

A spherical $t$-design is a spherical $\{t, t-1, \dots, 1\}$-design. A tight frame is just a spherical $\{2\}$-design. The study of spherical $T$-designs started from Bannai-Okuda-Tagami [BOT15]. Later Okuda-Yu [OY16] proved the nonexistence of tight spherical $\{4\}$-designs. Some further discussion of spherical $T$-designs can be found in [ZBBKY17].

## 3 Spherical embeddings of strongly regular graphs

In this section, we review the notion of spherical embeddings of strongly regular graphs and set up some necessary materials for following sections.

**Definition 3.1.** Let $\Gamma$ be a regular graph with $v$ vertices and valency $k$. Then $\Gamma$ is called *strongly regular* if every two adjacent vertices have $\lambda$ common neighbors and every two non-adjacent vertices have $\mu$ common neighbors. The tuple $(v, k, \lambda, \mu)$ is called the *type* of the strongly regular graph $\Gamma$.

Let $\Gamma$ be an SRG of type $(v, k, \lambda, \mu)$. The graph $\Gamma$ has three eigenvalues, one trivial eigenvalue $k$ with multiplicity 1, two nontrivial eigenvalues

$$r = \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right) \geq 0 \quad \text{and} \quad s = \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right) < 0$$

with multiplicities

$$m_r = \frac{1}{2}\left(v - 1 - \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right) \quad \text{and} \quad m_s = \frac{1}{2}\left(v - 1 + \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right),$$

respectively. Note that the eigenvalues $k$, $r$ and $s$ may not be distinct.

We call an SRG *primitive* if it is connected and its complement graph is connected as well. If $\Gamma$ is primitive, we always have $1 \leq k \leq v-2$, $\mu \geq 1$, $s < 0 < r < k$, $m_r \geq 1$ and $m_s \geq 1$.

An SRG is called *imprimitive* if it is not primitive. There are exactly two families of imprimitive SRGs, which are given in Example 3.2.

**Example 3.2.** Let $m$ and $k$ be positive integers.

(i) Let $mK_{k+1}$ be the disjoint union of $m$ copies of the complete graph $K_{k+1}$. It is an imprimitive SRG with $(r, s, m_r, m_s) = (k, -1, m-1, v-m)$, where $v - m \neq 0$ since $k$ is positive.

(ii) Let $\overline{mK_{k+1}}$ be the complement graph of $mK_{k+1}$. It is an imprimitive SRG with $(r, s, m_r, m_s) = (-k-1, 0, m-1, v-m)$, where $v - m \neq 0$ since $k$ is positive.

Brouwer-Cohen-Neumaier in [BCN89, Theorem 1.3.1] derived the following properties of the parameters of a primitive SRG:

$$\begin{cases} k = \mu - rs, \\ v = \frac{1}{\mu}(k-r)(k-s), \\ \lambda = r + s + \mu. \end{cases}$$

If $2k + (v-1)(\mu - \lambda) = 0$, then $\Gamma$ is called a *conference graph*. These are precisely the SRGs of type $(v, (v-1)/2, (v-5)/4, (v-1)/4)$ which have the same parameters as their complementary graphs. If $\Gamma$ is not a conference graph, then $r$ and $s$ are distinct integers with non-equal multiplicities $m_r$ and $m_s$, respectively.

In this paper, we want to study the spherical embeddings of $\Gamma$ with respect to $r$ and with respect to $s$. We will apply the same arguments to these two embeddings. Let $x$ and $y$ denote the nontrivial eigenvalues of an SRG of type $(v, k, \lambda, \mu)$, and denote by $m_x$ and $m_y$ the multiplicities of $x$ and $y$, respectively. In other words, $\{x, y\} = \{r, s\}$, but there is not a particular choice of $x$ and $y$.

For a primitive SRG, we have the following properties:

$$\begin{cases} k = \mu - xy, \\ v = \frac{1}{\mu}(k-x)(k-y), \\ \lambda = x + y + \mu. \end{cases} \tag{3.1}$$

Moreover, $x$ and $y$ are distinct real numbers satisfying $xy < 0$ and $x, y \neq k$. Substituting Eq. (3.1) into $(m_x, m_y) = (m_r, m_s)$ when $x > y$ and into $(m_x, m_y) = (m_s, m_r)$ when $x < y$, we obtain the following expressions for $m_x$ and $m_y$ when the SRG is primitive:

$$m_x = \frac{(\mu - xy)(\mu - xy - y)(y+1)}{\mu(y-x)} \text{ and } m_y = \frac{(\mu - xy)(\mu - xy - x)(x+1)}{\mu(x-y)}. \tag{3.2}$$

We identify $\Gamma$ with a symmetric association scheme $\mathfrak{X} = (\{\text{vertices of } \Gamma\}, \{R_i\}_{0 \le i \le 2})$ of class 2. More precisely, let the adjacency matrices of $R_0$, $R_1$ and $R_2$ be $I$, $A$ and $J - I - A$, respectively, where $A$ is the adjacency matrix of $\Gamma$, $I$ is the identity matrix and $J$ is the all one matrix. The SRG $\Gamma$ is primitive if and only if its corresponding association scheme $\mathfrak{X}$ is primitive.

Let $V_y$ be the eigenspace of $A$ with respect to the eigenvalue $y$ and $E_y$ the matrix representation of the orthogonal projection $\mathbb{R}^v \to V_y$. Denote the rank $m_y$ of $E_y$ by $n$ conventionally.

The *spherical embedding* of $\Gamma$ with respect to eigenvalue $y$ into the sphere $S^{n-1}$ is the map

$$\begin{aligned} \iota_y : \{\text{vertices of } \Gamma\} &\to S^{n-1}, \\ u &\mapsto \sqrt{\frac{v}{n}} E_y \psi_u, \end{aligned}$$

where $\psi_u \in \mathbb{R}^v$ is the characteristic vector of the vertex $u$. Let

$$\iota_y(\Gamma) := \{\iota_y(u) \mid u \text{ is a vertex of } \Gamma\}.$$

We will also call $\iota_y(\Gamma)$ the spherical embedding of $\Gamma$ with respect to the eigenvalue $y$. Note that this embedding exists only when $m_y \geq 1$.

**Example 3.3.** Let $m$ and $k$ be positive integers, and let $\Gamma$ be either the imprimitive SRG $mK_{k+1}$ or the imprimitive SRG $\overline{mK_{k+1}}$. The parameters of $\Gamma$ are given in Example 3.2. For the eigenvalue $r$, it is easy to check that the spherical embedding $\iota_r(\Gamma)$ either does not exist when $m = 1$, or is a 1-distance set when $m \geq 2$. For the eigenvalue $s$, by checking Eq. (2.1), it is easy to show that the spherical embedding $\iota_s(\Gamma)$ is not a spherical $\{4\}$-design.

From now on, we assume that $\Gamma$ is primitive. Recall that the second eigenmatrix $Q$ of $\mathfrak{X}$ (see [BI84]) has rows indexed by classes $\{0, 1, 2\}$ of $\mathfrak{X}$ and has columns indexed by eigenvalues $\{k, x, y\}$ of $\Gamma$. For any two vertices $u$ and $w$ in $\Gamma$, the inner product of $\iota_y(u)$ and $\iota_y(w)$ is calculated below.

$$\langle \iota_y(u), \iota_y(w) \rangle = \begin{cases} 1, & \text{if } u = w, \\ \frac{Q_y(1)}{n}, & \text{if } (u,w) \in R_1, \\ \frac{Q_y(2)}{n}, & \text{if } (u,w) \in R_2. \end{cases}$$

Then, $A(\iota_y(\Gamma)) = \left\{\frac{Q_y(1)}{n}, \frac{Q_y(2)}{n}\right\}$, where $Q_y(j)$ is the $(j,y)$-entry of the second eigenmatrix $Q$. Let $P_j(y)$ be the $(y,j)$-entry of the first eigenmatrix, equivalently, $P_j(y)$ denotes the eigenvalue of $R_j$ on $V_y$. Let $k_j$ be the valency of $R_j$. According to the relation between the first eigenmatrix and the second eigenmatrix, we have

$$A(\iota_y(\Gamma)) = \left\{\frac{P_1(y)}{k_1}, \frac{P_2(y)}{k_2}\right\} = \left\{\frac{y}{k}, \frac{-y-1}{v-k-1}\right\}. \tag{3.3}$$

**Proposition 3.4** ([DGS77, Theorem 4.8])**.** *Let $X$ be an $s$-distance set on the unit sphere $S^{n-1}$. Then $|X| \leq \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$.*

This implies that $v = |\iota_y(\Gamma)| \leq \frac{n(n+3)}{2}$. Moreover, Delsarte–Goethals–Seidel proved in [DGS77] that the spherical embedding of an SRG is a spherical 2-design. Then by Eq. (2.1) we can express the condition that the spherical embedding of primitive $\Gamma$ is a spherical $\{4\}$-design as

$$F := Q_{n,4}(1) + kQ_{n,4}\left(\frac{y}{k}\right) + (v-k-1)Q_{n,4}\left(\frac{-y-1}{v-k-1}\right) = 0, \tag{3.4}$$

where $Q_{n,4}(\xi) = \frac{n(n+6)}{24}\left((n^2+6n+8)\xi^4 - 6(n+2)\xi^2 + 3\right)$. Recall that $n = m_y$. Rewriting $F$ in terms of $x, y, \mu$ using Eqs. (3.1) and (3.2), when $\mu \neq 0$ we get

$$F = F_0(x,y,\mu)F_1(x,y,\mu)F_2(x,y,\mu)F_3(x,y,\mu),$$

where

$$\begin{aligned}
F_0(x,y,\mu) :=& \frac{(\mu - xy - x)(\mu - xy - y)^2}{24\mu^4(x+1)^2(x-y)^4(\mu - xy)^2}, && (3.5)\\
F_1(x,y,\mu) :=& (x+1)\mu^2 - \left(2(x^2+x+3)y + x(x-5)\right)\mu + x^2(x+1)y(y+1), && (3.6)\\
F_2(x,y,\mu) :=& (x+1)\mu^2 - \left(2(x^2+x+2)y + x(x-3)\right)\mu + x^2(x+1)y(y+1), && (3.7)\\
F_3(x,y,\mu) :=& (x+1)\left(-y + x(x^2+3x+3)\right)\mu^3 \\
& - \left((3x^2+8x+3)y^2 + xy(3x^4+10x^3+6x^2-7x-2) + x^3(x+2)(x+3)\right)\mu^2 \\
& + (x+1)y\left((3x^2-2x-2)y^2 + x(3x^4+5x^3-4x^2+x+1)y + x^4(2x+5)\right)\mu \\
& + x^2(x+1)^2y^2(y+1)(y-x^3). && (3.8)
\end{aligned}$$

**Proposition 3.5.** *Let $\Gamma$ be a primitive SRG. If the spherical embedding of $\Gamma$ with respect to the eigenvalue $y$ is a spherical $\{4,2,1\}$-design, then $F_i(x,y,\mu) = 0$ for some $i \in \{0,1,2,3\}$.*

*Proof.* Since the spherical embedding of an SRG will be a spherical 2-distance set and a spherical 2-design, so we only need to check the spherical $\{4\}$-design condition which is equivalent to condition (3.4), namely, $F_i(x,y,\mu) = 0$ for some $i \in \{0,1,2,3\}$. ☐

Let $\overline{\Gamma}$ be the *complement graph* of $\Gamma$. Then $\overline{\Gamma}$ is again an SRG of type $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$ whose eigenvalues are $v-k-1, -x-1, -y-1$. Hence condition (2.1) implies the following result.

**Lemma 3.6.** *Let $\Gamma$ be a primitive SRG, and let $T$ be a set of positive integers. The spherical embedding $\iota_y(\Gamma)$ is a spherical $T$-design if and only if the spherical embedding $\iota_{-y-1}(\overline{\Gamma})$ is a spherical $T$-design.*

*Proof.* One can check that

$$A(\iota_{-y-1}(\overline{\Gamma})) = \left\{ \frac{-y-1}{v-k-1}, \frac{y}{k} \right\}.$$

Then $\iota_{-y-1}(\overline{\Gamma})$ is a spherical $T$-design if and only if

$$Q_{n,t}(1) + (v-k-1)Q_{n,t}\left(\frac{-y-1}{v-k-1}\right) + kQ_{n,t}\left(\frac{y}{k}\right) = 0 \quad \text{for all } t \in T.$$

Note that the equality above is also the condition for $\iota_y(\Gamma)$ being a spherical $T$-design. $\square$

# 4 Proof of the main result

The purpose of this section is to prove our main result Theorem 1.1. An important step is to analyze the integer zeros of the $F_i(x, y, \mu)$'s in Eqs. (3.5) to (3.8).

**Proposition 4.1.** *If $xy < 0$, $\mu > 0$, $\mu - xy - x \neq 0$ and $\mu - xy - y \neq 0$, then the rational functions $F_0(x, y, \mu)$, $F_1(x, y, \mu)$ and $F_2(x, y, \mu)$ have no integer zeros.*

*Proof.* It is straightforward to see that $F_0(x, y, \mu)$ have no integer zeros. When $x \geq 1$ and $y \leq -1$, we have

$$\begin{aligned} F_1(x, y, \mu) &\geq -\left(2(x^2+x+3)y + x(x-5)\right)\mu \geq (x+6)(x+1)\mu > 0, \\ F_2(x, y, \mu) &\geq -\left(2(x^2+x+2)y + x(x-3)\right)\mu \geq (x+4)(x+1)\mu > 0, \end{aligned}$$

and when $x \leq -1$ and $y \geq 1$

$$\begin{aligned} F_1(x, y, \mu) &\leq -\left(2(x^2+x+3)y + x(x-5)\right)\mu \leq -3(x^2-x+2)\mu < 0, \\ F_2(x, y, \mu) &\leq -\left(2(x^2+x+2)y + x(x-3)\right)\mu \leq -(3x^2-x+4)\mu < 0. \end{aligned}$$

This completes the proof. $\square$

The integer zeros of $F_3(x, y, \mu)$ are much harder to find. We list in Table 4.1 all the integer zeros $(x, y, \mu)$, together with some related parameters, for $1 \leq x \leq 5$ and $-1000 \leq y \leq -1$.

We observe from Table 4.1 that, except the first column (where $n = 2$), all integer zeros belong to two infinite parametric families in Table 4.2. This observation motivates us to give the following conjecture.

**Conjecture 4.2.** All integer zeros of $F_3(x, y, \mu)$ with $x > 0$, $y < 0$ and $\mu > 0$ belong to Table 4.2.

We will prove a partial result of this conjecture in Theorem 5.1, where we assume $v \leq n(n+3)/2$ in addition. The proof of Theorem 5.1 will be postponed to § 5, since it is a standalone result and it is extremely technical.

| $x$ | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-1$ | $-5$ | $-5$ | $-28$ | $-28$ | $-81$ | $-81$ | $-176$ | $-176$ | $-325$ | $-325$ |
| $n = m_y$ | 2 | 6 | 7 | 22 | 23 | 46 | 47 | 78 | 79 | 118 | 119 |
| $v$ | 3 | 27 | 28 | 275 | 276 | 1127 | 1128 | 3159 | 3160 | 7139 | 7140 |
| $k$ | 2 | 10 | 15 | 112 | 140 | 486 | 567 | 1408 | 1584 | 3250 | 3575 |
| $\lambda$ | 1 | 1 | 6 | 30 | 58 | 165 | 246 | 532 | 708 | 1305 | 1630 |
| $\mu$ | 1 | 5 | 10 | 56 | 84 | 243 | 324 | 704 | 880 | 1625 | 1950 |

Table 4.1: Small integer zeros of $F_3(x, y, \mu)$ with positive $x$. Related parameters are obtained by Eqs. (3.1) and (3.2).

| $x$ | 1 | $t$ | $t$ |
|---|---|---|---|
| $y$ | $-1$ | $-t^2(2t+3)$ | $-t^2(2t+3)$ |
| $n = m_y$ | 2 | $4t^2+4t-2$ | $4t^2+4t-1$ |
| $v$ | 3 | $(2t+1)^2(2t^2+2t-1)$ | $2t(t+1)(4t^2+4t-1)$ |
| $k$ | 2 | $2t^3(2t+3)$ | $t^2(2t+1)(2t+3)$ |
| $\lambda$ | 1 | $t(2t-1)(t^2+t-1)$ | $t(2t^3+3t^2+1)$ |
| $\mu$ | 1 | $t^3(2t+3)$ | $t^2(t+1)(2t+3)$ |

Table 4.2: Conjectural integer zeros of $F_3(x, y, \mu)$. Related parameters are obtained by Eqs. (3.1) and (3.2).

*Proof of Theorem 1.1.* Since $X$ is a spherical 2-distance 2-design, $X$ carries the structure of a symmetric association scheme of class 2. Then, $X$ is a spherical embedding of an SRG $\Gamma$ with respect to some eigenvalue $y$ of $\Gamma$ into the sphere $S^{n-1}$ for some positive integer $n$, namely $X = \iota_y(\Gamma)$. If $\Gamma$ is imprimitive, then by Example 3.3, either $X$ does not exist, or $X$ is spherical 1-distance set, or $X$ is not a spherical $\{4\}$-design. Therefore, $\Gamma$ is primitive.

Let $(v, k, \lambda, \mu)$ be the type of $\Gamma$, where $v$, $k$ and $\mu$ are all integers, and let $k$, $x$ and $y$ be the three eigenvalues of $\Gamma$. Since $\Gamma$ is primitive, we have $xy < 0$, $\mu \geq 1$, $m_x \neq 0$, which implies that $\mu - xy - y \neq 0$ by Eq. (3.2), and $m_y \neq 0$, which implies that $\mu - xy - x \neq 0$ by Eq. (3.2).

Since $X$ is a spherical 2-distance set, we have an upper bound $v = |X| \leq n(n+3)/2$.

Since $X$ is a spherical $\{4\}$-design, we have the lower bound $|X| \geq \frac{(n+1)(n+2)}{6}$. If $\Gamma$ is a conference graph, then $2n + 1 = |X| \geq \frac{(n+1)(n+2)}{6}$, which implies that $n \leq 9$. We list all the conference graphs with $2 \leq n \leq 9$ in Table 4.3.

| $v$ | 5 | 9 | 13 | 17 |
|---|---|---|---|---|
| $k$ | 2 | 4 | 1 | 2 |
| $\lambda$ | 0 | 1 | 2 | 3 |
| $\mu$ | 1 | 2 | 3 | 4 |

Table 4.3: Conference graphs with $5 \leq v \leq 19$.

We can check that among the four types of conference graphs in Table 4.3, only the spherical embedding of conference graph of type $(5,2,0,1)$ is a spherical $\{4\}$-design. Moreover, its spherical embedding is a pentagon on $S^1$, which is a tight 4-design. From now on, we assume that $\Gamma$ is not a conference graph, hence $x$ and $y$ are integers.

Now let us summarize all numerical conditions on parameters we get.

- $x$, $y$, $\mu$, $n$ and $v$ are all integers.
- $xy<0$ and $\mu \geq 1$.
- $\mu - xy - x \neq 0$ and $\mu - xy - y \neq 0$.
- $v \leq n(n+3)/2$.

**Case 1**. $y \leq -1$.

By Proposition 3.5, we know that $F_i(x,y,\mu)=0$ for at least one $i \in \{0,1,2,3\}$. Proposition 4.1 shows that there are no such integer solutions when $i \in \{0,1,2\}$, and Theorem 5.1 shows that all integer solutions of $F_3(x,y,\mu)=0$ are:

(i) $(1,-1,1)$;

(ii) $(t,-t^2(2t+3),t^3(2t+3))$ for positive integers $t$;

(iii) $(t,-t^2(2t+3),t^2(2t+3)(t+1))$ for positive integers $t$.

Therefore, $(x,y,\mu)$ belongs to one of the three types of solutions above.

**Case 1.1**. *$(x,y,\mu)$ is a type (i) solution.*

In this case, $n=2$ and $X$ is a regular triangle on $S^1$, which is not a spherical 2-distance set.

**Case 1.2**. *$(x,y,\mu)$ is a type (ii) solution.*

Using Eqs. (3.1) and (3.2), we write parameters $n$, $v$, $k$ and $\lambda$ in $t$, as shown in Table 4.2. Then, Eq. (3.3) becomes

$$A(X)=\left\{-\frac{1}{2t},\frac{1}{2(t+2)}\right\}.$$

By writing every paremeter in terms of $t$, one can check that we have

$$Q_{n,3}(1)+kQ_{n,3}\left(-\frac{1}{2t}\right)+(v-k-1)Q_{n,3}\left(\frac{1}{2(t+2)}\right)=0,$$

which means that $X$ is a spherical $\{3\}$-design, in addition to the assumption that $X$ is a spherical $\{4,2,1\}$-design. Therefore, $X$ is a spherical 4-design. Moreover, it is easy to check that $|X|=v=n(n+3)/2$, which implies that $X$ is a tight spherical 4-design on $S^{n-1}$.

**Case 1.3**. *$(x,y,\mu)$ is a type (iii) solution.*

Similarly, using Eqs. (3.1) and (3.2), we write related parameters in $t$, as shown in Table 4.2, and by Eq. (3.3),

$$A(X)=\left\{\pm\frac{1}{2t+1}\right\}.$$

Since $t$ is positive, $-1 \notin A(X)$, hence $X$ and $-X$ are disjoint set. The disjoint union $X' := X \cup (-X)$ of spherical $\{4,2,1\}$-desings is also a $\{4,2,1\}$-design. Since $X'$ is antipodal,

$X'$ is a spherical $\{5,3,1\}$-design. Therefore, $X'$ is a spherical 5-design on $S^{n-1}$. It is easy to check that $|X| = v = n(n+1)/2$, which implies that $X'$ is a tight spherical 5-design on $S^{n-1}$.

**Case 2**. $y \geq 0$.

Let $\overline{\Gamma}$ be the complement graph of $\Gamma$, and let $Y := \iota_{-y-1}(\overline{\Gamma})$. The spherical embedding $Y$ is a spherical 2-distance set since $\overline{\Gamma}$ is a primitive SRG, and $Y$ is also a spherical $\{4,2,1\}$-design by Lemma 3.6. Moreover, its eigenvalue $-y-1$ is negative. Applying the result in Case 1 to $Y$, we know that either $Y$ is a tight spherical 4-design, or $Y \cup -Y$ is a tight spherical 5-design. Applying Lemma 3.6 to $\overline{\Gamma}$, we have that either $X$ is a tight spherical 4-design, or $X \cup -X$ is a tight spherical 5-design.

Therefore, the proof of Theorem 1.1 completes. □

# 5 Integer zeros of $F_3$

The main result of this section is Theorem 5.1, which gives a partial result of Conjecture 4.2 and is used in the proof of Theorem 1.1 in § 4. Note that the proof of Theorem 5.1 does not use any results in all previous sections.

The proof of Theorem 5.1 relies on computer calculations heavily. We first explain and demonstrate in § 5.1 how we use computers in the proof, and then give the proof of Theorem 5.1 in § 5.2.

## 5.1 Computer calculations

In this paper, we only ask computers to do two kinds of calculations:

(i) prove polynomial inequality for "large" real variables;

(ii) solve polynomial equations for "small" integer variables.

There are well-established algorithms to do (i), for instance cylindrical algebraic decomposition, and (ii) only requires enumeration of finitely many "small" possible tuples. In theory, both (i) and (ii) can be done by hand. In practice, since we human do not have as much computational power as computers do, we can only do (i) for "very large" real variables, say analysis of asymptotic behavior, and do (ii) for "very small" integer variables.

Note that Hilbert's 10-th problem shows that there are no algorithms to solve general Diophantine equations. We do not ask computers to find out all integer solutions $F_3(x, y, \mu) = 0$ for us directly. We only use computers as an extended calculator.

Before giving the proof of Theorem 5.1, we demonstrate explicitly how we use computers for a much simpler polynomial $F_1(x, y, \mu)$. Recall that

$$F_1(x, y, \mu) = (x+1)\mu^2 - \big(2(x^2+x+3)y + x(x-5)\big)\,\mu + x^2(x+1)y(y+1),$$

and let us consider the problem of finding all integer solutions of $F_1(x, y, \mu) = 0$ with $xy \leq -1$, $\mu \geq 1$.

We first use computers to do the following three things.

(i) Prove that $F_1(x,y,\mu)>0$ if $x\geq 1$, $y\leq -1$ and $\mu\geq 1$.

(ii) Prove that $F_1(x,y,\mu)<0$ if $x\leq -1$, $y\geq 1$ and $\mu\geq 1$.

(iii) For each integral tuple $(x,y,\mu)$ that is not in the above two cases, test if $F_1(x,y,\mu)=0$.

In Mathematica, the command "Simplify[Expression, Assumption]" will return "True" if the expression hold under the assumption. We use the command

$$\text{Simplify}\Big[(x+1)\mu^2-\big(2(x^2+x+3)y+x(x-5)\big)\mu+x^2(x+1)y(y+1)>0, x\geq 1\&\&y\leq -1\&\&\mu\geq 1\Big]$$

to do (i), and use the command

$$\text{Simplify}\Big[(x+1)\mu^2-\big(2(x^2+x+3)y+x(x-5)\big)\mu+x^2(x+1)y(y+1)<0, x\leq -1\&\&y\geq 1\&\&\mu\geq 1\Big]$$

to do (ii). There are only finitely many (in fact, zero) tuples in (iii), hence we can simply ask computers to enumerate all the tuples. We find no integer zeros in this case.

We then analyze the results obtained by computers. All integer tuples $(x,y,\mu)$ satisfying the assumption are covered by (i), (ii) and (iii). If the tuple is considered in either (i) or (ii), then the computer calculation shows that either $F_1(x,y,\mu)>0$, or $F_1(x,y,\mu)<0$, hence $(x,y,\mu)$ is not a zero. If the tuple is considered in (iii), then it can not be a zero since the computer calculation finds no integer zeros in (iii). Therefore, we can conclude that $F_1(x,y,\mu)$ has no integer zeros with $xy\leq -1$ and $\mu\geq 1$.

## 5.2 Analysis of $F_3$

We employ the strategy of [Xia18b] to find all integer solutions of $F_3$ in a specific region satisfying some additional integer conditions. The equation $F_3=0$ gives a surface in a three dimensional space. We first analyze the asymptotic behavior of all real points on the surface. Then, we construct a good surface $z=0$ that is sufficiently close to surface $F_3=0$ under integer conditions, so that all large integer points on the surface $F_3=0$ are on the new surface $z=0$ as well. Thus, we reduce the problem of finding integer points on the two dimensional surface $F_3=0$ to the problem of finding integer points on a one dimensional curve, the intersection of $F_3=0$ and $z=0$. Then, we repeat this procedure and reduce the dimension of the problem, until we find all the large solutions.

All variables and numbers in this section are assumed to be real, unless stated explicitly otherwise. Many arguments in the proof of Theorem 5.1 require us to distinguish variables and numbers. So, we put subscripts, say $_0$, for numbers. For instance, $x$ and $y^{(1)}$ are real variables, and $x_0$ and $y_0^{(1)}$ are real numbers.

In this section, we regard $x$, $y$ and $\mu$ as variables, and use a different but equivalent definition for $n$ and $v$. Let

$$n := \frac{(x+1)(\mu-xy)(\mu-xy-x)}{\mu(-y+x)} \in \mathbb{Q}(x,y,\mu), \tag{5.1}$$

$$v := n+1-\frac{yn+\mu-xy}{x} \in \mathbb{Q}(x,y,\mu). \tag{5.2}$$

For an arbitrary variable $t$ in this section, we set $t_0$ to be the specialization of $t$ to $x=x_0$ and possibly $y=y_0$, $\mu=\mu_0$, etc. For instance $n_0 := n|_{x=x_0,y=y_0,\mu=\mu_0}$ and $v_0 := v|_{x=x_0,y=y_0,\mu=\mu_0}$.

**Theorem 5.1.** *Consider the region*

$$D := \{(x_0, y_0, \mu_0) \in \mathbb{R}^3 \mid x_0 \geq 1, y_0 \leq -1, \mu_0 \geq 1, v_0 \leq n_0(n_0+3)/2\}.$$

*Then, all integer solutions* $(x_0, y_0, \mu_0)$ *of* $F_3(x, y, \mu) = 0$ *in* $D$ *such that* $n_0$ *and* $v_0$ *are integers are given in Table 5.1.*

| | | | |
|---|---|---|---|
| $x_0$ | $1$ | $t_0$ | $t_0$ |
| $y_0$ | $-1$ | $-t_0^2(2t_0+3)$ | $-t_0^2(2t_0+3)$ |
| $\mu_0$ | $1$ | $t_0^3(2t_0+3)$ | $t_0^2(t_0+1)(2t_0+3)$ |

Table 5.1: One special solution and two parametric solutions, where $t_0$ is a positive integer.

*Proof.* In this proof, all computer calculations are done in Mathematica. The complete Mathematica code used is available in [Xia18a].

**Step 1** Use computers to prove that if $x \geq 1$,

$$y \in (-\infty, -(2x^3+3x^2+3x+2)] \cup [-(2x^3+3x^2-3x-3), -1], \tag{5.3}$$

and $\mu \geq 1$, then either $v > n(n+3)/2$ or $F_3(x, y, \mu) > 0$.

**Step 2** Let $a$ be defined by

$$\mu = -(x+a)y. \tag{5.4}$$

We substitute Eq. (5.4) into $F_3(x, y, \mu) \in \mathbb{Z}[x, y, \mu]$ and set

$$G_1(a; x, y) := F_3(x, y, \mu)/y^2 \in \mathbb{Z}[a][x, y].$$

Use computers to prove that:

**Step 2**(a) $G_1(a; x, 0) < 0$ if $x \geq 2$ and $a \in (-x, +\infty)$.

**Step 2**(b) $G_1(a; x, -1) > 0$ if $x \geq 2$ and $a \in (-x, +\infty)$.

**Step 2**(c) $G_1(a; x, -(2x^3+3x^2+3x+2)) > 0$ if $x \geq 2$ and $a \in (-x, -1] \cup [3, +\infty)$.

In this step, let $a_0$ and $x_0$ be some real numbers such that $x_0 \geq 2$ and $a_0 \in (-x_0, +\infty)$. Then, $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is a degree $\leq 2$ polynomial in $y$.

By **Step 2**(a) and **Step 2**(b), the polynomial $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has a solution

$$y_1 \in (-1, 0). \tag{5.5}$$

When the coefficient of $y^2$ in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is positive, by **Step 2**(a), $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has a solution

$$y_2 \in (0, +\infty). \tag{5.6}$$

When the coefficient of $y^2$ in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is negative and $a_0 \in (-x, -1] \cup [3, +\infty)$, by **Step 2**(c), $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has a solution

$$y_2 \in (-\infty, -(2x_0^3+3x_0^2+3x_0+2)). \tag{5.7}$$

Therefore, when the coefficient of $y^2$ in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is nonzero and $a_0 \in (-x, -1] \cup [3, +\infty)$, the degree 2 polynomial $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has at most two solutions and we have found two different solutions, one in Eq. (5.5) and the other in either Eq. (5.6) or Eq. (5.7), hence all solutions of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ are $y_1$ and $y_2$.

When the coefficient of $y^2$ in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is zero, $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has at most one (in fact, exactly one) solution and we have found one solution in Eq. (5.5), hence $y_1$ is the unique solution.

Therefore, if $a_0 \in (-x, -1] \cup [3, +\infty)$, then all solutions of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ are in

$$(-\infty, -(2x_0^3 + 3x_0^2 + 3x_0 + 2)) \cup (-1, 0) \cup (0, +\infty), \tag{5.8}$$

and if $a_0 \in [-1, 3]$, then there exists at most one solution in $(-\infty, -1)$.

**Step 3** In this step, let $(x_0, y_0, \mu_0)$ be a tuple in $D$ such that $x_0 \geq 90$ and $F_3(x_0, y_0, \mu_0) = 0$. By Eq. (5.3) in **Step 1**,

$$y_0 \in (-(2x_0^3 + 3x_0^2 + 3x_0 + 2), -(2x_0^3 + 3x_0^2 - 3x_0 - 3)). \tag{5.9}$$

Let $a_0$ be defined using $x_0$, $y_0$, $\mu_0$ and Eq. (5.4). Since $\mu_0 \geq 1$, $a_0 \in (-x_0, +\infty)$. If $a_0 \in (-x_0, -1] \cup [3, +\infty)$, then by Eq. (5.8) in **Step 2**,

$$y_0 \in (-\infty, -(2x_0^3 + 3x_0^2 + 3x_0 + 2)) \cup (-1, 0) \cup (0, +\infty). \tag{5.10}$$

Clearly, Eqs. (5.9) and (5.10) contradicts with each other, which implies that $a_0 \in [-1, 3]$.

Let $b$ be defined by

$$\begin{aligned} -y =& 2x^3 + 3x^2 + \frac{3(a-1)a}{2}x - \frac{3(a-1)^2 a}{2} \\ &+ \frac{3(a-1)a\,(3a^2 - 4a + 2)}{4}x^{-1} - \frac{3(a-1)a^2\,(4a^2 - 6a + 3)}{4}x^{-2} \\ &+ \frac{3(a-1)a\,(11a^4 - 20a^3 + 16a^2 - 9a + 5)}{8}x^{-3} + bx^{-4}. \end{aligned} \tag{5.11}$$

We substitute Eq. (5.11) into $G_1(a; x, y) \in \mathbb{Z}[a](x, y)$ and set

$$G_2(a, b; x) := G_1(a; x, y) \in \mathbb{Q}[a, b][x, x^{-1}].$$

Use computers to prove that:

**Step 3**(a) $G_2(a, -3994; x) > 0$ if $x \geq 90$ and $a \in [-1, 3]$.

**Step 3**(b) $G_2(a, 64; x) < 0$ if $x \geq 90$ and $a \in [-1, 3]$.

Consider the polynomial $G_2(a_0, b; x_0) \in \mathbb{R}[b]$. By **Step 3**(a) and **Step 3**(b), the polynomial $G_2(a_0, b; x_0) \in \mathbb{R}[b]$ has a solution $b_3$ in $[-3994, 64]$. Let $b_0$ be defined using $a_0$, $x_0$ and $y_0$ by Eq. (5.11). Clearly, $b_0$ is also a solution of $G_2(a_0, b; x_0) \in \mathbb{R}[b]$. Since $G_2(a_0, b; x_0) \in \mathbb{R}[b]$ is of degree 2, we cannot conclude immediately that $b_0 \in$

$[-3994, 64]$, and we need to use the uniqueness of the solutions of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ to prove it.

Let $y_3$ be defined using $a_0$, $x_0$ and $b_3$ by Eq. (5.11). Then $y_3$ is a solution of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$. Using the bound on $a_0$, $x_0$ and $b_3$, we get $y_3 \in (-\infty, -1)$. Recall from Eq. (5.9) that $y_0 \in (-\infty, -1)$ is a solution of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$. **Step 2** proves that $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has at most one solution in $(-\infty, -1)$ and we have found two solutions $y_0, y_3 \in (-\infty, -1)$, hence $y_3 = y_0$. Therefore, $b_0 = b_3 \in [-3994, 64]$.

**Step 4** Let

$$m^2 := n - (4x^2 + 4x - 2), \tag{5.12}$$

where $m$ is a complex variable such that Re $m \geq 0$ and $m^2$ is real. We substitute Eqs. (5.4) and (5.11) into Eq. (5.12) and set

$$G_3(a, b; x) := m^2 \in \mathbb{Q}[a, b](x).$$

Consider a new complex variable $\widetilde{m}$ such that Re $\widetilde{m} \geq 0$ and $\widetilde{m}^2$ is real, and let $c$ be defined by

$$\begin{aligned}\widetilde{m}^2 =& a^2 - (a-1)a^2x^{-1} + (a-1)\left(a^2+1\right)ax^{-2} - \frac{(a-1)\left(2a^3+2a+1\right)a}{2}x^{-3} \\ &+ \frac{(a-1)a\left(4a^4+4a^2-a+7\right)}{4}x^{-4} + cx^{-5},\end{aligned} \tag{5.13}$$

We use the right side of Eq. (5.13) to define $G_4$:

$$G_4(a, c; x) := \widetilde{m}^2 \in \mathbb{Q}[a, c][x^{-1}].$$

Note that we will regard $m^2$ and $\widetilde{m}^2$ as the same after specialization, namely $m_0^2 = \widetilde{m}_0^2$. The only reason why we introduce $\widetilde{m}^2$ is that, we want to distinguish the uses of $m^2$ as $G_3$ and $m^2$ as $G_4$ in **Step 5**.

Use computers to prove that:

**Step 4**(a) $G_3(a, b; x) > G_4(a, -1620; x)$ if $x \geq 90$, $a \in [-1, 3]$ and $b \in [-3994, 64]$.

**Step 4**(b) $G_3(a, b; x) < G_4(a, 3; x)$ if $x \geq 90$, $a \in [-1, 3]$ and $b \in [-3994, 64]$.

**Step 4**(c) $G_4(a, c; x) < 9$ if $x \geq 90$, $a \in [-1, 3]$ and $c \in [-1620, 3]$.

In this step, let $(x_0, y_0, \mu_0)$ be a tuple in $D$ such that $x_0 \geq 90$ and $F_3(x_0, y_0, \mu_0) = 0$. Let $a_0$ be defined using $x_0$, $y_0$ and $\mu_0$ by Eq. (5.4). Then, by the discussion in **Step 3**, we get $a_0 \in [-1, 3]$. Let $b_0$ be defined using $a_0$, $x_0$ and $y_0$ by Eq. (5.11). The conclusion of **Step 3** shows that $b_0 \in [-3994, 64]$.

Now, consider the equation $\mathbb{R} \ni G_3(a_0, b_0; x_0) = G_4(a_0, c; x_0) \in \mathbb{R}[c]$. It is a linear equation in $c$, hence there exists a unique solution $c_0$. **Step 4**(a) and **Step 4**(b) prove that $c_0 \in [-1620, 3]$.

Let $\widetilde{m}_0^2$ be defined using $a_0$, $x_0$ and $c_0$ by Eq. (5.13). Then, **Step 4**(c) means $m_0^2 = \widetilde{m}_0^2 < 9$.

**Step 5** Let

$$z := 144m^2 - \left(3v + \left(8y + 4x^3 + 6x^2 + 3\right)(2x+1) - \frac{3m^2(m^2-7)}{2}\right)^2, \tag{5.14}$$

and

$$\widetilde{z} := 144\widetilde{m}^2 - \left(3\widetilde{v} + \left(8y + 4x^3 + 6x^2 + 3\right)(2x+1) - \frac{3\widetilde{m}^2(\widetilde{m}^2-7)}{2}\right)^2. \tag{5.15}$$

where

$$\widetilde{v} := \widetilde{n} + 1 - \frac{y\widetilde{n} + \mu - xy}{x} \quad \text{and} \quad \widetilde{n} := \widetilde{m}^2 + 4x^2 + 4x - 2. \tag{5.16}$$

We substitute Eqs. (5.4), (5.11) and (5.13) into Eq. (5.15) and set

$$G_5(a,b,c;x) := \widetilde{z} \in \mathbb{Q}[a,b,c][x^{-1}].$$

Thus, we can express $G_5$ as

$$G_5(a,b,c;x) = \sum_{i=-20}^{-2} G_{5,i}x^i,$$

for some $G_{5,i} \in \mathbb{Q}[a,b,c]$. Use computers do the following things:

**Step 5**(a) Give a good upper bound on $G_{5,i}$ when $a \in [-1,3]$, $b \in [-3994, 64]$ and $c \in [-1620, 3]$.

**Step 5**(b) Use **Step 5**(a) to prove that $|G_5(a,b,c;x)| < 1$ if $x \geq 120$, $a \in [-1,3]$, $b \in [-3994, 64]$ and $c \in [-1620, 3]$.

In this step, let $(x_0, y_0, \mu_0)$ be a tuple in $D$ such that $x_0$, $y_0$, $\mu_0$, $n_0$, $v_0$ are all integers, $x_0 \geq 120$ and $F_3(x_0, y_0, \mu_0) = 0$. By Eq. (5.14), we know that $z_0$ is an integer.

Consider an additional equation $m_0^2 = \widetilde{m}_0^2$, which implies that $\widetilde{n}_0 = n_0$ by Eqs. (5.1) and (5.16), $\widetilde{v}_0 = v_0$ by Eqs. (5.2) and (5.16) and $\widetilde{z}_0 = z_0$ by Eqs. (5.14) and (5.15). **Step 4** shows that Eq. (5.4) defines an $a_0 \in [-1,3]$, Eq. (5.11) defines a $b_0 \in [-3994, 64]$ and the equation $m_0^2 = \widetilde{m}_0^2$ defines a $c_0 \in [-1620, 3]$. Then, **Step 5**(a) and **Step 5**(b) prove that $|\widetilde{z}_0| < 1$.

Since $z_0 = \widetilde{z}_0$, we know that $z_0$ is an integer and $|z_0| < 1$. Therefore, $z_0 = 0$. According to Eq. (5.14), $144m_0^2 = 144m_0^2 - z_0$ is the square of an integer. Thus, $m_0$ is an integer. Since Re $m_0 \geq 0$, $m_0$ is an nonnegative integer. **Step 4** shows that $m_0^2 < 9$, then we have $m_0 \in \{0, 1, 2\}$.

**Step 6** In **Step 6** and **Step 7**, we assume that $m = m_0$ for some $m_0 \in \{0,1,2\}$, and the goal of these two steps is to find out suitable solutions of the system of equations $F_3(x,y,\mu) = 0$ and $m = m_0$.

Consider the expression

$$\mu(y-x)(m^2 - m_0^2). \tag{5.17}$$

We substitute Eqs. (5.1) and (5.12) into Eq. (5.17) and set

$$G_6(m_0; x, y; \mu) := \mu(y-x)(m^2 - m_0^2) \in \mathbb{Z}[m_0][x, y][\mu].$$

Regarding polynomials $G_6(m_0; x, y, \mu) \in \mathbb{Z}[m_0][x, y][\mu]$ and $F_3(x, y, \mu) \in \mathbb{Z}[x, y][\mu]$ as polynomials in a single variable $\mu$, we apply the extended Euclidean algorithm to them and get

$$F_3(x, y, \mu)p(m_0; x, y)+G_6(m_0; x, y; \mu)q(m_0; x, y) = x^2(x+1)y^2(y+1)^3(y-x)^2F_4(m_0; x, y) \tag{5.18}$$

for some nonzero minimal polynomials $p(m_0; x, y), q(m_0; x, y), F_4(m_0; x, y) \in \mathbb{Z}[m_0][x, y]$. Moreover, the choice for $F_4(m_0; x, y)$ is unique up to sign. We use the convention that $F_4(m_0; x, y)$ is the unique nonzero minimal polynomial satisfying Eq. (5.18) such that the coefficient of $x^{13}$ in $F_4(m_0; x, y)$ is positive.

**Step 7** In this step, we find all large solutions of $F_4(m_0; x, y) = 0$ in a certain region, for $m_0 \in \{0, 1, 2\}$. The polynomial $F_4(m_0; x, y)$ is of degree 3 in $y$. Let

$$\begin{aligned}
y^{(1)} &:= -\left(2x^3 + 3x^2 + \frac{3m_0(m_0+1)}{2}x + \frac{3m_0(m_0+1)}{4}\right),\\
y^{(2)} &:= -\left(2x^3 + 3x^2 + \frac{3m_0(m_0-1)}{2}x + \frac{3m_0(m_0-1)}{4}\right),\\
y^{(3)} &:= x.
\end{aligned}$$

Note that when $x \geq 3$ and $m_0 = 0$, we have

$$y^{(1)} < y^{(2)} + \frac{1}{x} < y^{(2)} + \frac{1}{2} < y^{(3)} - 1 < y^{(3)} < y^{(3)} + 1 \tag{5.19}$$

and when $x \geq 1$ and $m_0 \in \{1, 2\}$, we have

$$y^{(1)} - \frac{1}{2} < y^{(1)} < y^{(1)} + \frac{1}{2} < y^{(2)} - \frac{1}{2} < y^{(2)} < y^{(2)} + \frac{1}{2} < y^{(3)} - 1 < y^{(3)} < y^{(3)} + 1. \tag{5.20}$$

Use computers to prove that:

**Step 7**(a) $F_4(m_0; x, y^{(1)}) = 0$ if $m_0 = 0$.

**Step 7**(b) $F_4(m_0; x, y^{(1)} - \frac{1}{2}) > 0$ if $x \geq 90$ and $m_0 \in [1, 2]$.

**Step 7**(c) $F_4(m_0; x, y^{(1)} + \frac{1}{2}) < 0$ if $x \geq 90$ and $m_0 \in [1, 2]$.

**Step 7**(d) $F_4(m_0; x, y^{(2)} + \frac{1}{x}) < 0$ if $x \geq 90$ and $m_0 = 0$.

**Step 7**(e) $F_4(m_0; x, y^{(2)} - \frac{1}{2}) < 0$ if $x \geq 90$ and $m_0 \in [1, 2]$.

**Step 7**(f) $F_4(m_0; x, y^{(2)} + \frac{1}{2}) > 0$ if $x \geq 90$ and $m_0 \in [0, 2]$.

**Step 7**(g) $F_4(m_0; x, y^{(3)} - 1) > 0$ if $x \geq 1$ and $m_0 \in [0, 2]$.

**Step 7**(h) $F_4(m_0; x, y^{(3)} + 1) < 0$ if $x \geq 1$ and $m_0 \in [0, 2]$.

Let $x_0 \geq 90$ be an integer. Consider the polynomial $F_4(m_0; x_0, y)$ in single variable $y$ of degree 3.

When $m_0 = 0$, **Step 7**(a) gives a solution $y_0^{(1)} \in (y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2})$, **Step 7**(d) and **Step 7**(f) gives a solution in $(y_0^{(2)} + \frac{1}{x}, y_0^{(2)} + \frac{1}{2}) \subset (y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2})$, **Step 7**(g) and **Step 7**(h) gives a solution in $(y_0^{(3)} - 1, y_0^{(3)} + 1)$, and these three solutions are all different by Eq. (5.19). Therefore, when $m_0 = 0$, all solutions of $F_4(m_0; x_0, y)$ are in

$$(y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2}) \cup (y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2}) \cup (y_0^{(3)} - 1, y_0^{(3)} + 1). \tag{5.21}$$

When $m_0 \in \{1, 2\}$, **Step 7**(b) and **Step 7**(c) give a solution in $(y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2})$, **Step 7**(e) and **Step 7**(f) give a solution in $(y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2})$, **Step 7**(g) and **Step 7**(h) give a solution in $(y_0^{(3)} - 1, y_0^{(3)} + 1)$, and these three solutions are all different by Eq. (5.20). Therefore, when $m_0 \in \{1, 2\}$, all solutions of $F_4(m_0; x_0, y)$ are also in Eq. (5.21).

Now, let $y_0 \leq -1$ be an integer such that $F_4(m_0; x_0, y_0) = 0$. By the discussion above, we know that $y_0$ is in

$$(y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2}) \cup (y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2}) \cup (y_0^{(3)} - 1, y_0^{(3)} + 1). \tag{5.22}$$

Since $m_0 \in \{0, 1, 2\}$, all of $y_0^{(1)}$, $y_0^{(2)}$ and $y_0^{(3)}$ are in $\frac{1}{2}\mathbb{Z}$. Therefore, Eq. (5.22) contains a unique negative integer $-x_0^2(2x_0 + 3)$ when $m_0 \in \{0, 1\}$ and no integers when $m_0 = 2$.

Therefore, for $m_0 \in \{0, 1, 2\}$, all integer solutions $(x_0, y_0)$ of $F_4(m_0; x, y) = 0$ such that $x_0 \geq 90$ and $y_0 \leq -1$ are $(t_0, -t_0^2(2t_0 + 3))$ for positive integer $t_0$.

**Step 8** Use computers to prove that, for all integer $x_0 \in [1, 120]$ and all negative integer $y_0 \in [-(2x_0^3 + 3x_0^2 + 3x_0 + 2), -(2x_0^3 + 3x_0^2 - 3x_0 - 3] \setminus \{-x_0^2(2x_0 + 3)\}$, the polynomial $F_3(x_0, y_0, \mu)$ in $\mu$ has a positive integer solution if and only if $x_0 = 1$ and $y_0 = -1$. Moreover, $F_3(1, -1, \mu)$ has a unique positive integer solution $\mu_0 = 1$.

**Step 9** It is easy to check all tuples in Table 5.1 are solutions of $F_3(x, y, \mu)$.

Let $(x_0, y_0, \mu_0)$ be a tuple in $D$ such that $n_0$ and $v_0$ are integers and $F_3(x_0, y_0, \mu_0) = 0$.

By Eq. (5.3) in **Step 1**, $y_0 \in [-(2x_0^3 + 3x_0^2 + 3x_0 + 2), -(2x_0^3 + 3x_0^2 - 3x_0 - 3)]$. Then, by **Step 8**, either $x_0 \geq 120$, or $(x_0, y_0, \mu_0) = (1, -1, 1)$ which is in Table 5.1, or $y_0 = -x_0^2(2x_0 + 3)$. For the last case, there are exactly three solutions of $F_3(x_0, y_0, \mu) = 0$ in $\mu$: $-x_0 y_0$ which is in Table 5.1, $-(x_0 + 1)y_0$ which is also in Table 5.1, and $-2x_0^4 - x_0^3 + x_0^2$ which is negative. Therefore, Table 5.1 contains all solutions when $x_0 \leq 120$. Now, assume that $x_0 \geq 120$.

By **Step 5**, $m_0 \in \{0, 1, 2\}$. Then, **Step 6** shows that $F_4(m_0; x_0, y_0) = 0$. **Step 7** solves $F_4(m_0; x_0, y_0) = 0$ and gives $y_0 = -x_0^2(2x_0 + 3)$. By the discussion above, all solutions with $y_0 = -x_0^2(2x_0 + 3)$ are in Table 5.1.

Therefore, all solutions are found and the proof of Theorem 5.1 is finished. □

## Remarks from Nozaki

After we finish the proof, we receive a remark from Hiroshi Nozaki who suggests us that there exists a short approach to our main result. Let $X$ be a spherical 2-distance $\{4, 2, 1\}$-design on $S^{n-1}$. If $X$ is a $\{3\}$-design, then $X$ is a spherical 2-distance 4-design, hence by [DGS77] it is a tight spherical 4-design. If $X$ is not a spherical $\{3\}$-design, then $X$ is a spherical 2-distance 2-design but not a 3-design. Therefore, in the latter case, by [Neu81] or [NS11], we have $|X| \leq \frac{n(n+1)}{2}$. Consider the multiset $X' := X \cup -X$, then $|X'| \leq n(n+1)$. Since $X'$ is antipodal, $X'$ is a $\{5, 3\}$-design. In conjunction with $\{4, 2, 1\}$-design assumption, $X'$ is a spherical 5-design, which has the lower bound $|X'| \geq n(n+1)$ by [DGS77]. Therefore, $X'$ is a tight spherical 5-design.

Above discussion covers our main result, but we want to indicate that our method is important and has potential to be applied to other combinatorics problems since there are a lot of combinatoric problems which can be reduced to the problems of solving the integer solutions of a Diophantine equation.

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