Classification of feasible parameters by solving Diophantine equations

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Abstract

In algebraic combinatorics, the first step of the classification of interesting objects is usually to find all their feasible parameters. The feasible parameters are often integral solutions of some complicated Diophantine equations, which cannot be solved by known methods. In this paper, we develop a method to solve such Diophantine equations in 3 variables. We demonstrate it by giving a classification of finite subsets that are spherical 2-distance sets and spherical $\{4, 2, 1\}$-designs at the same time.

1 Introduction

1.1 Diophantine equations and classification problems

The feasible parameters of interesting algebraic combinatorial objects are often solutions of some complicated Diophantine equations. Since the Hilbert's tenth problem has a negative answer, it is not practical to solve a Diophantine equation using only the information of the equation itself. However, due to the combinatorial nature of the feasible parameters, we usually have some additional integer conditions and positivity conditions, for example, some rational functions in the feasible parameters are integers or positive. Describing all integral solutions of a Diophantine equation with these additional conditions can be easier sometimes.

In this paper, we develop a method to solve Diophantine equations in 3 variables under such additional conditions. We demonstrate it by giving a classification of spherical 2-distance $\{4, 2, 1\}$-designs in Theorem 1.1. We first construct a related Diophantine equation, and find as many integral conditions and positivity conditions as possible. Then, we solve it using our method. At the end, for each solution, we give a description of the spherical 2-distance $\{4, 2, 1\}$-designs having the solution as feasible parameters.

1.2 Spherical 2-distance $\{4, 2, 1\}$-designs

The study of *spherical designs* and *spherical codes* can be tracked from Delsarte-Goethals-Seidel [13]. They proved that $t \leq 2s$ for any spherical s-distance t-design. It is also shown that if $t \geq 2s - 2$, then a spherical t-designs carries a structure of an s-class Q-polynomial association scheme. The classification of spherical s-distance t-designs with $t \geq 2s-1$ is an interesting problem. It is known that $t = 2s$ if and only if X is a tight spherical $2s$-design. When X is antipodal, $t = 2s-1$ if and only if X is a tight spherical $(2s-1)$-design. The classification of tight spherical designs is still an open problem [1]. There are many works done for spherical 2-distance 3-designs [9, 11, 12]. There are too many sporadic feasible parameters for such subsets, and for almost all feasible parameters, the existence of the corresponding 2-distance 3-designs is unknown.

Introduced by Bannai-Okuda-Tagami [6], *spherical T-designs* for a set of integers T are generalizations of spherical t-designs. Their definition will be given in § 2.1. Barg et al. studied in [8] a finite subset on the unit sphere which is a 2-distance set and a spherical $\{2\}$-design (i.e. tight frame) at the same time. It

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is proved in [8, Theorem 1] that such a finite subset is either a spherical embedding of a strongly regular graph, or a shifted spherical embedding of a strongly regular graph, or an equiangular tight frame.

The main result of this paper is a classification of finite sets X which are spherical 2-distance sets and spherical $\{4,2,1\}$-designs at the same time. It is known in [13] that since X is a spherical 2-distance $\{2,1\}$-design, X is a spherical embedding of a strongly regular graph. Moreover, X being a spherical $\{4\}$-design allows us to classify all the possible strongly regular graphs involved and give the following classification result.

Theorem 1.1. *Let $X \subset S^{n-1}$ be a finite subset where $n \geq 2$. Suppose that X is a spherical 2-distance $\{4,2,1\}$-design. Then, one of the following holds:*

(i) *X is a tight spherical 4-design on S^{n-1};*

(ii) *the disjoint union $X \cup (-X)$ is a tight spherical 5-design on S^{n-1}.*

We first analyze the behavior of spherical embeddings of strongly regular graphs in § 2.3, and construct a related Diophantine equation in three variables. Then, we employ our method to solve it in Theorem 4.1. The final proof of Theorem 1.1 is given in § 3. Note that the proof of Theorem 4.1 does not rely on any results in previous sections. We put it in § 4 since we want to isolate the part of the proof involving our new method to solve Diophantine equation.

Remark 1.2. (i) In Theorem 1.1(ii), X can be regarded as a half of a tight spherical 5-design. However, it is unknown if there exists a good way to choose a half of a given tight spherical 5-design to get an X. More precisely, it might be a difficult problem to choose a point from each antipodal pair of points in a given tight spherical 5-design to get a subset that is a spherical 2-distance $\{4,2,1\}$-design. Partly motivated by this result, we studied in [7] when a half of an antipodal spherical t-design becomes a spherical 1-design.

(ii) In Theorem 1.1, tight spherical 4-designs and tight spherical 5-designs appear. The classification of tight spherical designs has been studied for several decades [13, 2, 3, 5, 14], but tight spherical t-designs for $t \in \{4,5,7\}$ are not classified yet.

1.3 Spherical 3-distance 5-designs

Many people have conjectured, but not written down, the classification of 3-distance 5-designs. Based on numerical experiments, we have Conjecture 1.3. We believe that our proof of Theorem 1.1 can be generalized and provide a possible method to attack Conjecture 1.3.

Conjecture 1.3. Let $X \subset S^{n-1}$ be a finite subset where $n \geq 2$. If X is a spherical 3-distance 5-design, then one of the following holds:

(i) $n = 2$ and X is the vertices of a regular hexagon or a regular heptagon;

(ii) X is a tight spherical 5-design on S^{n-1};

(iii) X is a section of a tight spherical 7-design on S^{n-1}.

In case (iii), there are four distinct inner product values $\{-1, 0, \pm\alpha\}$ between distinct points in the tight spherical 7-design. A *section* means the collection of points that have the same inner product value α with a fixed point in the design.

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2 Preliminary

2.1 Spherical T-design

Let $S^{n-1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ be the real unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$. A finite subset X of the unit sphere is an *s-distance set* if there are exactly s distinct distances (equivalently, inner products) between two distinct points in X, namely the set

$$A(X) := \{\langle x, y\rangle \mid x, y \in X, x \neq y\},$$

has cardinality s, where $\langle\ ,\ \rangle$ is the usual Euclidean inner product.

Let T be a set of some positive integers. A finite subset $X \subset S^{n-1}$ is called a *spherical design of harmonic index T*, or simply *spherical T-design*, on S^{n-1} if

$$\sum_{x\in X} f(x) = 0$$

holds for every homogeneous harmonic polynomial f of degree t with $t \in T$. It is well-known that X is a spherical T-design on S^{n-1} if and only if

$$\sum_{x,y\in X} Q_{n,t}(\langle x, y\rangle) = 0 \quad \text{for every } t \in T, \tag{2.1}$$

where $Q_{n,t}(\xi)$ is the Gegenbauer polynomial of degree t in one variable ξ. In this paper, $Q_{n,t}(\xi)$ is normalized so that $Q_{n,t}(1) = \binom{n+t-1}{n-1} - \binom{n+t-3}{n-1}$. This value equals the dimension of the vector space of all homogeneous harmonic polynomials of degree t in n variables.

A spherical t-design is a spherical $\{t, t-1, \dots, 1\}$-design. The study of spherical T-designs started from Bannai-Okuda-Tagami [6]. Later Okuda-Yu [17] proved the nonexistence of tight spherical $\{4\}$-designs. Some further discussion of spherical T-designs can be found in [20].

2.2 Strongly regular graphs

In this section, we review the notion of strongly regular graphs.

Definition 2.1. Let Γ be a regular graph with v vertices and valency k. Then Γ is called *strongly regular* if every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. The tuple (v, k, λ, μ) is called the *type* of the strongly regular graph Γ.

Let Γ be a strongly regular graph of type (v, k, λ, μ). The graph Γ has three eigenvalues, one trivial eigenvalue k with multiplicity 1, two nontrivial eigenvalues

$$r = \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right) \geq 0 \quad \text{and} \quad s = \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right) < 0$$

with multiplicities

$$m_r = \frac{1}{2}\left(v - 1 - \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right) \quad \text{and} \quad m_s = \frac{1}{2}\left(v - 1 + \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right),$$

respectively. Note that the eigenvalues k, r and s may not be distinct.

We call a strongly regular graph *primitive* if it is connected and its complement graph is connected as well. If Γ is primitive, we always have $1 \leq k \leq v-2$, $\mu \geq 1$, $s < 0 < r < k$, $m_r \geq 1$ and $m_s \geq 1$.

An strongly regular graphs is called *imprimitive* if it is not primitive. There are exactly two families of imprimitive strongly regular graphss, which are given in Example 2.2.

Example 2.2. Let m and k be positive integers.

(i) Let mK_{k+1} be the disjoint union of m copies of the complete graph K_{k+1}. It is an imprimitive strongly regular graphs with $(r, s, m_r, m_s) = (k, -1, m-1, km)$, where $m_s > 0$.

(ii) Let $\overline{mK_{k+1}}$ be the complement graph of mK_{k+1}. It is an imprimitive strongly regular graphs with $(r, s, m_r, m_s) = (-k-1, 0, m-1, km)$, where $m_s > 0$.

Brouwer-Cohen-Neumaier in [10, Theorem 1.3.1] derived the following properties of the parameters of a primitive strongly regular graphs:

$$\begin{cases} k = \mu - rs, \\ v = \frac{1}{\mu}(k-r)(k-s), \\ \lambda = r + s + \mu. \end{cases}$$

If $2k + (v-1)(\mu - \lambda) = 0$, then Γ is called a *conference graph*. These are precisely the strongly regular graphss of type $(v, (v-1)/2, (v-5)/4, (v-1)/4)$ which have the same parameters as their complementary graphs. If Γ is not a conference graph, then r and s are distinct integers with non-equal multiplicities m_r and m_s, respectively.

In this paper, we want to study the spherical embeddings of Γ with respect to r and with respect to s. We will apply the same arguments to these two embeddings. Let x and y denote the nontrivial eigenvalues of a strongly regular graph of type (v, k, λ, μ), and denote by m_x and m_y the multiplicities of x and y, respectively. In other words, $\{x, y\} = \{r, s\}$, but there is not a particular choice of x and y.

For a primitive strongly regular graphs, we have the following properties:

$$\begin{cases} k = \mu - xy, \\ v = \frac{1}{\mu}(k-x)(k-y), \\ \lambda = x + y + \mu. \end{cases} \tag{2.2}$$

Moreover, x and y are distinct real numbers satisfying $xy < 0$ and $x, y \neq k$. Substituting Eq. (2.2) into $(m_x, m_y) = (m_r, m_s)$ when $x > y$ and into $(m_x, m_y) = (m_s, m_r)$ when $x < y$, we obtain the following expressions for m_x and m_y when the strongly regular graphs is primitive:

$$m_x = \frac{(\mu - xy)(\mu - xy - y)(y+1)}{\mu(y-x)} \text{ and } m_y = \frac{(\mu - xy)(\mu - xy - x)(x+1)}{\mu(x-y)}. \tag{2.3}$$

We identify Γ with a symmetric association scheme $\mathfrak{X} = (\{\text{vertices of } \Gamma\}, \{R_i\}_{0 \le i \le 2})$ of class 2. More precisely, let the adjacency matrices of R_0, R_1 and R_2 be I, A and $J - I - A$, respectively, where A is the adjacency matrix of Γ, I is the identity matrix and J is the all one matrix. The strongly regular graphs Γ is primitive if and only if its corresponding association scheme $\mathfrak{X}$ is primitive.

Let V_y be the eigenspace of A with respect to the eigenvalue y and E_y the matrix representation of the orthogonal projection $\mathbb{R}^v \to V_y$. Denote the rank m_y of E_y by n conventionally.

The *spherical embedding* of Γ with respect to eigenvalue y into the sphere S^{n-1} is the map

$$\begin{aligned} \iota_y : \{\text{vertices of } \Gamma\} &\to S^{n-1}, \\ u &\mapsto \sqrt{\frac{v}{n}} E_y \psi_u, \end{aligned}$$

where $\psi_u \in \mathbb{R}^v$ is the characteristic vector of the vertex u. Let

$$\iota_y(\Gamma) := \{\iota_y(u) \mid u \text{ is a vertex of } \Gamma\}.$$

We will also call $\iota_y(\Gamma)$ the spherical embedding of Γ with respect to the eigenvalue y. Note that this embedding exists only when $m_y \geq 1$.

Example 2.3. Let m and k be positive integers, and let Γ be either the imprimitive strongly regular graphs mK_{k+1} or the imprimitive strongly regular graphs $\overline{mK_{k+1}}$. The parameters of Γ are given in Example 2.2. For the eigenvalue r, it is easy to check that the spherical embedding $\iota_r(\Gamma)$ either does not exist when $m = 1$, or is a 1-distance set when $m \geq 2$. For the eigenvalue s, by checking Eq. (2.1), it is easy to show that the spherical embedding $\iota_s(\Gamma)$ is not a spherical $\{4\}$-design.

2.3 Spherical embeddings of strongly regular graphs

From now on, we assume that Γ is a primitive strongly regular graph. Recall that the second eigenmatrix Q of $\mathfrak{X}$ (see [4]) has rows indexed by classes $\{0,1,2\}$ of $\mathfrak{X}$ and has columns indexed by eigenvalues $\{k,x,y\}$ of Γ. For any two vertices u and w in Γ, the inner product of $\iota_y(u)$ and $\iota_y(w)$ is calculated below.

$$\langle \iota_y(u), \iota_y(w)\rangle = \begin{cases} 1, & \text{if } u = w, \\ \frac{Q_y(1)}{n}, & \text{if } (u,w) \in R_1, \\ \frac{Q_y(2)}{n}, & \text{if } (u,w) \in R_2. \end{cases}$$

Then, $A(\iota_y(\Gamma)) = \left\{\frac{Q_y(1)}{n}, \frac{Q_y(2)}{n}\right\}$, where $Q_y(j)$ is the (j,y)-entry of the second eigenmatrix Q. Let $P_j(y)$ be the (y,j)-entry of the first eigenmatrix, equivalently, $P_j(y)$ denotes the eigenvalue of R_j on V_y. Let k_j be the valency of R_j. According to the relation between the first eigenmatrix and the second eigenmatrix, we have

$$A(\iota_y(\Gamma)) = \left\{\frac{P_1(y)}{k_1}, \frac{P_2(y)}{k_2}\right\} = \left\{\frac{y}{k}, \frac{-y-1}{v-k-1}\right\}. \tag{2.4}$$

Proposition 2.4 ([13, Theorem 4.8])**.** *Let X be an s-distance set on the unit sphere S^{n-1}. Then $|X| \le \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1}$.*

This implies that $v = |\iota_y(\Gamma)| \le \frac{n(n+3)}{2}$. Moreover, Delsarte–Goethals–Seidel proved in [13] that the spherical embedding of a strongly regular graph is a spherical 2-design. Then by Eq. (2.1) we can express the condition that the spherical embedding of primitive Γ is a spherical $\{4\}$-design as $F = 0$ where

$$F := Q_{n,4}(1) + kQ_{n,4}\left(\frac{y}{k}\right) + (v-k-1)Q_{n,4}\left(\frac{-y-1}{v-k-1}\right) \tag{2.5}$$

and $Q_{n,4}(\xi) = \frac{n(n+6)}{24}\left((n^2+6n+8)\xi^4 - 6(n+2)\xi^2 + 3\right)$. Recall that $n = m_y$. Rewriting F in terms of x, y, μ using Eqs. (2.2) and (2.3), when $\mu \neq 0$ we get

$$F = F_0(x,y,\mu)F_1(x,y,\mu)F_2(x,y,\mu)F_3(x,y,\mu),$$

where

$$\begin{aligned}
F_0(x,y,\mu) :=& \frac{(\mu - xy - x)(\mu - xy - y)^2}{24\mu^4(x+1)^2(x-y)^4(\mu - xy)^2}, && (2.6)\\
F_1(x,y,\mu) :=& (x+1)\mu^2 - \left(2(x^2+x+3)y + x(x-5)\right)\mu + x^2(x+1)y(y+1), && (2.7)\\
F_2(x,y,\mu) :=& (x+1)\mu^2 - \left(2(x^2+x+2)y + x(x-3)\right)\mu + x^2(x+1)y(y+1), && (2.8)\\
F_3(x,y,\mu) :=& (x+1)\left(-y + x(x^2+3x+3)\right)\mu^3 \\
&- \left((3x^2+8x+3)y^2 + xy(3x^4+10x^3+6x^2-7x-2) + x^3(x+2)(x+3)\right)\mu^2 \\
&+ (x+1)y\left((3x^2-2x-2)y^2 + x(3x^4+5x^3-4x^2+x+1)y + x^4(2x+5)\right)\mu \\
&+ x^2(x+1)^2y^2(y+1)(y-x^3). && (2.9)
\end{aligned}$$

Proposition 2.5. *Let Γ be a primitive strongly regular graphs. If the spherical embedding of Γ with respect to the eigenvalue y is a spherical $\{4,2,1\}$-design, then $F_i(x,y,\mu) = 0$ for some $i \in \{0,1,2,3\}$.*

Proof. Since the spherical embedding of a strongly regular graph will be a spherical 2-distance set and a spherical 2-design, so we only need to check the spherical $\{4\}$-design condition which is equivalent to $F = 0$ for F in Eq. (2.5), namely, $F_i(x,y,\mu) = 0$ for some $i \in \{0,1,2,3\}$. $\square$

Let $\overline{\Gamma}$ be the *complement graph* of Γ. Then $\overline{\Gamma}$ is again a strongly regular graph of type $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$ whose eigenvalues are $v-k-1, -x-1, -y-1$. Hence, condition (2.1) implies the following result.

Lemma 2.6. *Let Γ be a primitive strongly regular graphs, and let T be a set of positive integers. The spherical embedding $\iota_y(\Gamma)$ is a spherical T-design if and only if the spherical embedding $\iota_{-y-1}(\overline{\Gamma})$ is a spherical T-design.*

Proof. One can check that

$$A(\iota_{-y-1}(\overline{\Gamma})) = \left\{ \frac{-y-1}{v-k-1}, \frac{y}{k} \right\}.$$

Then $\iota_{-y-1}(\overline{\Gamma})$ is a spherical T-design if and only if

$$Q_{n,t}(1) + (v-k-1)Q_{n,t}\left(\frac{-y-1}{v-k-1}\right) + kQ_{n,t}\left(\frac{y}{k}\right) = 0 \quad \text{for all } t \in T.$$

Note that the equality above is also the condition for $\iota_y(\Gamma)$ being a spherical T-design. Thus, the result follows. $\square$

3 The classification of spherical 2-distance $\{4, 2, 1\}$-designs

The purpose of this section is to prove Theorem 1.1. An important step is to analyze the integer zeros of the $F_i(x, y, \mu)$'s in Eqs. (2.6) to (2.9).

Proposition 3.1. *If $xy < 0$, $\mu > 0$, $\mu - xy - x \neq 0$ and $\mu - xy - y \neq 0$, then the rational functions $F_0(x, y, \mu)$, $F_1(x, y, \mu)$ and $F_2(x, y, \mu)$ have no integer zeros.*

Proof. It is straightforward to see that $F_0(x, y, \mu)$ have no integer zeros. When $x \geq 1$ and $y \leq -1$, we have

$$\begin{aligned} F_1(x, y, \mu) &\geq -\left(2(x^2+x+3)y + x(x-5)\right)\mu \geq (x+6)(x+1)\mu > 0, \\ F_2(x, y, \mu) &\geq -\left(2(x^2+x+2)y + x(x-3)\right)\mu \geq (x+4)(x+1)\mu > 0, \end{aligned}$$

and when $x \leq -1$ and $y \geq 1$

$$\begin{aligned} F_1(x, y, \mu) &\leq -\left(2(x^2+x+3)y + x(x-5)\right)\mu \leq -3(x^2-x+2)\mu < 0, \\ F_2(x, y, \mu) &\leq -\left(2(x^2+x+2)y + x(x-3)\right)\mu \leq -(3x^2-x+4)\mu < 0. \end{aligned}$$

This completes the proof. $\square$

The integer zeros of $F_3(x, y, \mu)$ are much harder to find. We list in Table 3.1 all the integer zeros (x, y, μ), together with some related parameters, for $1 \leq x \leq 5$ and $-1000 \leq y \leq -1$.

x	1	1	1	2	2	3	3	4	4	5	5
y	-1	-5	-5	-28	-28	-81	-81	-176	-176	-325	-325
$n = m_y$	2	6	7	22	23	46	47	78	79	118	119
v	3	27	28	275	276	1127	1128	3159	3160	7139	7140
k	2	10	15	112	140	486	567	1408	1584	3250	3575
λ	1	1	6	30	58	165	246	532	708	1305	1630
μ	1	5	10	56	84	243	324	704	880	1625	1950

Table 3.1: Small integer zeros of $F_3(x, y, \mu)$ with positive x. Related parameters are obtained by Eqs. (2.2) and (2.3).

We observe from Table 3.1 that, except the first column (where $n = 2$), all integer zeros belong to two infinite parametric families in Table 3.2. This observation motivates us to give the following conjecture.

Conjecture 3.2. All integer zeros of $F_3(x, y, \mu)$ with $x > 0$, $y < 0$ and $\mu > 0$ belong to Table 3.2.

We will prove a partial result of this conjecture in Theorem 4.1, where we assume $v \leq n(n+3)/2$ in addition. The proof of Theorem 4.1 will be postponed to § 4, since it is a standalone result and it is extremely technical.

x	1	t	t
y	-1	$-t^2(2t+3)$	$-t^2(2t+3)$
$n=m_y$	2	$4t^2+4t-2$	$4t^2+4t-1$
v	3	$(2t+1)^2(2t^2+2t-1)$	$2t(t+1)(4t^2+4t-1)$
k	2	$2t^3(2t+3)$	$t^2(2t+1)(2t+3)$
λ	1	$t(2t-1)\left(t^2+t-1\right)$	$t(2t^3+3t^2+1)$
μ	1	$t^3(2t+3)$	$t^2(t+1)(2t+3)$

Table 3.2: Conjectural integer zeros of $F_3(x,y,\mu)$. Related parameters are obtained by Eqs. (2.2) and (2.3).

Proof of Theorem 1.1. Since X is a spherical 2-distance 2-design, X carries the structure of a symmetric association scheme of class 2. Then, X is the spherical embedding of a strongly regular graph Γ with respect to some eigenvalue y of Γ into the sphere S^{n-1} for some positive integer n, namely $X=\iota_y(\Gamma)$. If Γ is imprimitive, then by Example 2.3, either X does not exist, or X is a spherical 1-distance set, or X is not a spherical $\{4\}$-design. Therefore, Γ is primitive.

Let (v,k,λ,μ) be the type of Γ, where v, k and μ are all integers, and let k, x and y be the three eigenvalues of Γ. Since Γ is primitive, we have $xy<0$, $\mu\geq 1$, $m_x\neq 0$, which implies that $\mu-xy-y\neq 0$ by Eq. (2.3), and $m_y\neq 0$, which implies that $\mu-xy-x\neq 0$ by Eq. (2.3).

Since X is a spherical 2-distance set, we have an upper bound $v=|X|\leq n(n+3)/2$.

Since X is a spherical $\{4\}$-design, we have the lower bound $|X|\geq\frac{(n+1)(n+2)}{6}$. If Γ is a conference graph, then $2n+1=|X|\geq\frac{(n+1)(n+2)}{6}$, which implies that $n\leq 9$. We list all the types of conference graphs with $2\leq n\leq 9$ in Table 3.3.

v	5	9	13	17
k	2	4	1	2
λ	0	1	2	3
μ	1	2	3	4

Table 3.3: Types of conference graphs with $5\leq v\leq 19$.

We can check that among the four types of conference graphs in Table 3.3, only the spherical embedding of the conference graph of type $(5,2,0,1)$ is a spherical $\{4\}$-design. Moreover, its spherical embedding is a pentagon on S^1, which is a tight spherical 4-design. From now on, we assume that Γ is not a conference graph, hence both x and y are integers.

Now let us summarize all numerical conditions on parameters we get.

- x, y, μ, n and v are all integers.
- $xy<0$ and $\mu\geq 1$.
- $\mu-xy-x\neq 0$ and $\mu-xy-y\neq 0$.
- $v\leq n(n+3)/2$.

Case 1. $y\leq -1$.

By Proposition 2.5, we know that $F_i(x,y,\mu)=0$ for at least one $i\in\{0,1,2,3\}$. Proposition 3.1 shows that there are no such integer solutions when $i\in\{0,1,2\}$, and Theorem 4.1 shows that all integer solutions of $F_3(x,y,\mu)=0$ are:

(i) $(1,-1,1)$;

(ii) $(t,-t^2(2t+3),t^3(2t+3))$ for positive integers t;

(iii) $(t,-t^2(2t+3),t^2(2t+3)(t+1))$ for positive integers t.

Therefore, (x, y, μ) belongs to one of the three types of solutions above.
Case 1.1. *(x, y, μ) is a type (i) solution.*

In this case, $n = 2$ and X is a regular triangle on S^1, which is not a spherical 2-distance set.
Case 1.2. *(x, y, μ) is a type (ii) solution.*

Using Eqs. (2.2) and (2.3), we write parameters n, v, k and λ in t, as shown in Table 3.2. Then, Eq. (2.4) becomes

$$A(X) = \left\{-\frac{1}{2t}, \frac{1}{2(t+2)}\right\}.$$

By writing every paremeter in terms of t, one can check that we have

$$Q_{n,3}(1) + kQ_{n,3}\left(-\frac{1}{2t}\right) + (v-k-1)Q_{n,3}\left(\frac{1}{2(t+2)}\right) = 0,$$

which means that X is a spherical $\{3\}$-design, in addition to the assumption that X is a spherical $\{4, 2, 1\}$-design. Therefore, X is a spherical 4-design. Moreover, it is easy to check that $|X| = v = n(n+3)/2$, which implies that X is a tight spherical 4-design on S^{n-1}.
Case 1.3. *(x, y, μ) is a type (iii) solution.*

Similarly, using Eqs. (2.2) and (2.3), we write related parameters in t, as shown in Table 3.2, and by Eq. (2.4),

$$A(X) = \left\{\pm\frac{1}{2t+1}\right\}.$$

Since t is positive, $-1 \notin A(X)$, hence X and $-X$ are disjoint sets. The disjoint union $X' := X \cup (-X)$ of spherical $\{4, 2, 1\}$-desings X and $-X$ is also a spherical $\{4, 2, 1\}$-design. Since X' is antipodal, X' is a spherical $\{5, 3, 1\}$-design. Therefore, X' is a spherical 5-design on S^{n-1}. It is easy to check that $|X| = v = n(n+1)/2$, which implies that X' is a tight spherical 5-design on S^{n-1}.
Case 2. *$y \geq 0$.*

Let $\overline{\Gamma}$ be the complement graph of Γ, and let $Y := \iota_{-y-1}(\overline{\Gamma})$. The spherical embedding Y is a spherical 2-distance set since $\overline{\Gamma}$ is a primitive strongly regular graphs, and Y is also a spherical $\{4, 2, 1\}$-design by Lemma 2.6. Moreover, its eigenvalue $-y-1$ is negative. Applying the result in Case 1 to Y, we know that either Y is a tight spherical 4-design, or $Y \cup (-Y)$ is a tight spherical 5-design. Applying Lemma 2.6 to $\overline{\Gamma}$, we have that either X is a tight spherical 4-design, or $X \cup (-X)$ is a tight spherical 5-design.

Therefore, the proof of Theorem 1.1 completes. □

4 Integer zeros of F_3

Let f be a generic polynomial in 3 variables with integer coefficients. We generalize the strategy of [19] to find all integer zeros of f in a specific region satisfying some additional integer conditions and positivity conditions. The equation $f = 0$ gives a surface in a three-dimensional space. We first analyze the asymptotic behavior of all real points on the surface. Then, we construct a good surface $z = 0$ such that (1) z takes integer values under the additional integer conditions, and (2) it is sufficiently close to surface $f = 0$ under the additional positivity conditions for large variables. Consequently, all large integer points on the surface $f = 0$ are on the new surface $z = 0$ as well. Thus, we reduce the problem of finding integer points on the two-dimensional surface $f = 0$ to the problem of finding integer points on a one dimensional curve, the intersection of $f = 0$ and $z = 0$. Then, we repeat this procedure and reduce the dimension of the problem, until we find all the large solutions. The small solutions are found by a computer search.

In this section, we apply this method to the Diophantine equation $F_3 = 0$. Theorem 4.1 gives a partial result of Conjecture 3.2 and is used in the proof of Theorem 1.1 in § 3. Note that the proof of Theorem 4.1 does not use any results in all previous sections. The proof of Theorem 4.1 relies on computer calculations heavily. We first explain in § 4.1 how we use computers in the proof, and then give the proof of Theorem 4.1 in § 4.2.

4.1 Computer calculations

In this paper, we only ask computers to do two kinds of calculations:

(i) prove polynomial inequality for "large" real variables;

(ii) solve polynomial equations for "small" integer variables.

There are well-established algorithms to do (i), for instance cylindrical algebraic decomposition, and (ii) only requires enumeration of finitely many "small" possible tuples. In theory, both (i) and (ii) can be done by hand. In practice, since we human do not have as much computational power as computers do, we can only do (i) for "very large" real variables, say analysis of asymptotic behavior, and do (ii) for "very small" integer variables.

Note that Hilbert's 10-th problem shows that there are no algorithms to solve general Diophantine equations. We do not ask computers to find out all integer solutions $F_3(x,y,\mu)=0$ for us directly. We only use computers as an extended calculator.

Before giving the proof of Theorem 4.1, we demonstrate explicitly how we use computers for a much simpler polynomial $F_1(x,y,\mu)$. Recall that

$$F_1(x,y,\mu)=(x+1)\mu^2-\big(2(x^2+x+3)y+x(x-5)\big)\,\mu+x^2(x+1)y(y+1),$$

and let us consider the problem of finding all integer solutions of $F_1(x,y,\mu)=0$ with $xy\le -1$, $\mu\ge 1$.

We first use computers to do the following three things.

(i) Prove that $F_1(x,y,\mu)>0$ if $x\ge 1$, $y\le -1$ and $\mu\ge 1$.

(ii) Prove that $F_1(x,y,\mu)<0$ if $x\le -1$, $y\ge 1$ and $\mu\ge 1$.

(iii) For each integral tuple (x,y,μ) that is not in the above two cases, test if $F_1(x,y,\mu)=0$.

In Mathematica, the command "Simplify[Expression, Assumption]" will return "True" if the expression hold under the assumption. We use the command

$$\text{Simplify}\big[(x+1)\mu^2-\big(2(x^2+x+3)y+x(x-5)\big)\mu+x^2(x+1)y(y+1)>0,x\geq1\&\&y\leq-1\&\&\mu\geq1\big]$$

to do (i), and use the command

$$\text{Simplify}\big[(x+1)\mu^2-\big(2(x^2+x+3)y+x(x-5)\big)\mu+x^2(x+1)y(y+1)<0,x\leq-1\&\&y\geq1\&\&\mu\geq1\big]$$

to do (ii). There are only finitely many (in fact, zero) tuples in (iii), hence we can simply ask computers to enumerate all the tuples. We find no integer zeros in this case.

We then analyze the results obtained by computers. All integer tuples (x,y,μ) satisfying the assumption are covered by (i), (ii) and (iii). If the tuple is considered in either (i) or (ii), then the computer calculation shows that either $F_1(x,y,\mu)>0$, or $F_1(x,y,\mu)<0$, hence (x,y,μ) is not a zero. If the tuple is considered in (iii), then it can not be a zero since the computer calculation finds no integer zeros in (iii). Therefore, we can conclude that $F_1(x,y,\mu)$ has no integer zeros with $xy\le -1$ and $\mu\ge 1$.

4.2 Analysis of F_3

All variables and numbers in this section are assumed to be real, unless stated explicitly otherwise. Many arguments in the proof of Theorem 4.1 require us to distinguish variables and numbers. So, we put subscripts, say $\ _0$, for numbers. For instance, x and $y^{(1)}$ are real variables, and x_0 and $y_0^{(1)}$ are real numbers.

In this section, we regard x, y and μ as variables, and use a different but equivalent definition for n and v. Let

$$n:=\frac{(x+1)(\mu-xy)(\mu-xy-x)}{\mu(-y+x)}\in\mathbb{Q}(x,y,\mu), \tag{4.1}$$

$$v:=n+1-\frac{yn+\mu-xy}{x}\in\mathbb{Q}(x,y,\mu). \tag{4.2}$$

For an arbitrary variable t in this section, we set t_0 to be the specialization of t to $x=x_0$ and possibly $y=y_0$, $\mu=\mu_0$, etc. For instance $n_0:=n|_{x=x_0,y=y_0,\mu=\mu_0}$ and $v_0:=v|_{x=x_0,y=y_0,\mu=\mu_0}$.

Theorem 4.1. *Consider the region*

$$D := \{(x_0, y_0, \mu_0) \in \mathbb{R}^3 \mid x_0 \geq 1, y_0 \leq -1, \mu_0 \geq 1, v_0 \leq n_0(n_0+3)/2\}.$$

Then, all integer solutions (x_0, y_0, μ_0) of $F_3(x, y, \mu) = 0$ in D such that n_0 and v_0 are integers are given in Table 4.1.

x_0	1	t_0	t_0
y_0	-1	$-t_0^2(2t_0+3)$	$-t_0^2(2t_0+3)$
μ_0	1	$t_0^3(2t_0+3)$	$t_0^2(t_0+1)(2t_0+3)$

Table 4.1: One special solution and two parametric solutions, where t_0 is a positive integer.

Proof. In this proof, all computer calculations are done in Mathematica. The complete Mathematica code used is available in [18].

Step 1 Use computers to prove that if $x \geq 1$,

$$y \in (-\infty, -(2x^3 + 3x^2 + 3x + 2)] \cup [-(2x^3 + 3x^2 - 3x - 3), -1], \tag{4.3}$$

and $\mu \geq 1$, then either $v > n(n+3)/2$ or $F_3(x, y, \mu) > 0$.

Step 2 Let a be defined by

$$\mu = -(x + a)y. \tag{4.4}$$

We substitute Eq. (4.4) into $F_3(x, y, \mu) \in \mathbb{Z}[x, y, \mu]$ and set

$$G_1(a; x, y) := F_3(x, y, \mu)/y^2 \in \mathbb{Z}[a][x, y].$$

Use computers to prove that:

Step 2(a) $G_1(a; x, 0) < 0$ if $x \geq 2$ and $a \in (-x, +\infty)$.

Step 2(b) $G_1(a; x, -1) > 0$ if $x \geq 2$ and $a \in (-x, +\infty)$.

Step 2(c) $G_1(a; x, -(2x^3 + 3x^2 + 3x + 2)) > 0$ if $x \geq 2$ and $a \in (-x, -1] \cup [3, +\infty)$.

In this step, let a_0 and x_0 be some real numbers such that $x_0 \geq 2$ and $a_0 \in (-x_0, +\infty)$. Then, $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is a degree ≤ 2 polynomial in y.

By **Step 2**(a) and **Step 2**(b), the polynomial $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has a solution

$$y_1 \in (-1, 0). \tag{4.5}$$

When the coefficient of y^2 in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is positive, by **Step 2**(a), $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has a solution

$$y_2 \in (0, +\infty). \tag{4.6}$$

When the coefficient of y^2 in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is negative and $a_0 \in (-x, -1] \cup [3, +\infty)$, by **Step 2**(c), $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has a solution

$$y_2 \in (-\infty, -(2x_0^3 + 3x_0^2 + 3x_0 + 2)). \tag{4.7}$$

Therefore, when the coefficient of y^2 in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is nonzero and $a_0 \in (-x, -1] \cup [3, +\infty)$, the degree 2 polynomial $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has at most two solutions and we have found two different solutions, one in Eq. (4.5) and the other in either Eq. (4.6) or Eq. (4.7), hence all solutions of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ are y_1 and y_2.

When the coefficient of y^2 in $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ is zero, $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has at most one (in fact, exactly one) solution and we have found one solution in Eq. (4.5), hence y_1 is the unique solution.

Therefore, if $a_0 \in (-x, -1] \cup [3, +\infty)$, then all solutions of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ are in

$$(-\infty, -(2x_0^3 + 3x_0^2 + 3x_0 + 2)) \cup (-1, 0) \cup (0, +\infty), \tag{4.8}$$

and if $a_0 \in [-1, 3]$, then there exists at most one solution in $(-\infty, -1)$.

Step 3 In this step, let (x_0, y_0, μ_0) be a tuple in D such that $x_0 \geq 90$ and $F_3(x_0, y_0, \mu_0) = 0$. By Eq. (4.3) in **Step 1**,

$$y_0 \in (-(2x_0^3 + 3x_0^2 + 3x_0 + 2), -(2x_0^3 + 3x_0^2 - 3x_0 - 3)). \tag{4.9}$$

Let a_0 be defined using x_0, y_0, μ_0 and Eq. (4.4). Since $\mu_0 \geq 1$, $a_0 \in (-x_0, +\infty)$. If $a_0 \in (-x_0, -1] \cup [3, +\infty)$, then by Eq. (4.8) in **Step 2**,

$$y_0 \in (-\infty, -(2x_0^3 + 3x_0^2 + 3x_0 + 2)) \cup (-1, 0) \cup (0, +\infty). \tag{4.10}$$

Clearly, Eqs. (4.9) and (4.10) contradicts with each other, which implies that $a_0 \in [-1, 3]$.

Let b be defined by

$$\begin{aligned} -y =& 2x^3 + 3x^2 + \frac{3(a-1)a}{2}x - \frac{3(a-1)^2 a}{2} \\ &+ \frac{3(a-1)a\left(3a^2 - 4a + 2\right)}{4}x^{-1} - \frac{3(a-1)a^2\left(4a^2 - 6a + 3\right)}{4}x^{-2} \\ &+ \frac{3(a-1)a\left(11a^4 - 20a^3 + 16a^2 - 9a + 5\right)}{8}x^{-3} + bx^{-4}. \end{aligned} \tag{4.11}$$

We substitute Eq. (4.11) into $G_1(a; x, y) \in \mathbb{Z}[a](x, y)$ and set

$$G_2(a, b; x) := G_1(a; x, y) \in \mathbb{Q}[a, b][x, x^{-1}].$$

Use computers to prove that:

Step 3(a) $G_2(a, -3994; x) > 0$ if $x \geq 90$ and $a \in [-1, 3]$.

Step 3(b) $G_2(a, 64; x) < 0$ if $x \geq 90$ and $a \in [-1, 3]$.

Consider the polynomial $G_2(a_0, b; x_0) \in \mathbb{R}[b]$. By **Step 3**(a) and **Step 3**(b), the polynomial $G_2(a_0, b; x_0) \in \mathbb{R}[b]$ has a solution b_3 in $[-3994, 64]$. Let b_0 be defined using a_0, x_0 and y_0 by Eq. (4.11). Clearly, b_0 is also a solution of $G_2(a_0, b; x_0) \in \mathbb{R}[b]$. Since $G_2(a_0, b; x_0) \in \mathbb{R}[b]$ is of degree 2, we cannot conclude immediately that $b_0 \in [-3994, 64]$, and we need to use the uniqueness of the solutions of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ to prove it.

Let y_3 be defined using a_0, x_0 and b_3 by Eq. (4.11). Then y_3 is a solution of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$. Using the bound on a_0, x_0 and b_3, we get $y_3 \in (-\infty, -1)$. Recall from Eq. (4.9) that $y_0 \in (-\infty, -1)$ is a solution of $G_1(a_0; x_0, y) \in \mathbb{R}[y]$. **Step 2** proves that $G_1(a_0; x_0, y) \in \mathbb{R}[y]$ has at most one solution in $(-\infty, -1)$ and we have found two solutions $y_0, y_3 \in (-\infty, -1)$, hence $y_3 = y_0$. Therefore, $b_0 = b_3 \in [-3994, 64]$.

Step 4 Let

$$m^2 := n - (4x^2 + 4x - 2), \tag{4.12}$$

where m is a complex variable such that $\operatorname{Re} m \geq 0$ and m^2 is real. We substitute Eqs. (4.4) and (4.11) into Eq. (4.12) and set

$$G_3(a, b; x) := m^2 \in \mathbb{Q}[a, b](x).$$

Consider a new complex variable $\widetilde{m}$ such that $\operatorname{Re} \widetilde{m} \geq 0$ and $\widetilde{m}^2$ is real, and let c be defined by

$$\begin{aligned} \widetilde{m}^2 =& a^2 - (a-1)a^2x^{-1} + (a-1)\left(a^2+1\right)ax^{-2} - \frac{(a-1)\left(2a^3 + 2a + 1\right)a}{2}x^{-3} \\ &+ \frac{(a-1)a\left(4a^4 + 4a^2 - a + 7\right)}{4}x^{-4} + cx^{-5}, \end{aligned} \tag{4.13}$$

We use the right side of Eq. (4.13) to define G_4:

$$G_4(a, c; x) := \widetilde{m}^2 \in \mathbb{Q}[a, c][x^{-1}].$$

Note that we will regard m^2 and $\widetilde{m}^2$ as the same after specialization, namely $m_0^2 = \widetilde{m}_0^2$. The only reason why we introduce $\widetilde{m}^2$ is that, we want to distinguish the uses of m^2 as G_3 and m^2 as G_4 in **Step 5**.

Use computers to prove that:

Step 4(a) $G_3(a,b;x) > G_4(a,-1620;x)$ if $x \geq 90$, $a \in [-1,3]$ and $b \in [-3994,64]$.

Step 4(b) $G_3(a,b;x) < G_4(a,3;x)$ if $x \geq 90$, $a \in [-1,3]$ and $b \in [-3994,64]$.

Step 4(c) $G_4(a,c;x) < 9$ if $x \geq 90$, $a \in [-1,3]$ and $c \in [-1620,3]$.

In this step, let (x_0, y_0, μ_0) be a tuple in D such that $x_0 \geq 90$ and $F_3(x_0, y_0, \mu_0) = 0$. Let a_0 be defined using x_0, y_0 and μ_0 by Eq. (4.4). Then, by the discussion in **Step 3**, we get $a_0 \in [-1,3]$. Let b_0 be defined using a_0, x_0 and y_0 by Eq. (4.11). The conclusion of **Step 3** shows that $b_0 \in [-3994,64]$.

Now, consider the equation $\mathbb{R} \ni G_3(a_0,b_0;x_0) = G_4(a_0,c;x_0) \in \mathbb{R}[c]$. It is a linear equation in c, hence there exists a unique solution c_0. **Step 4**(a) and **Step 4**(b) prove that $c_0 \in [-1620,3]$.

Let $\widetilde{m}_0^2$ be defined using a_0, x_0 and c_0 by Eq. (4.13). Then, **Step 4**(c) means $m_0^2 = \widetilde{m}_0^2 < 9$.

Step 5 Let

$$z := 144m^2 - \left(3v + \left(8y + 4x^3 + 6x^2 + 3\right)(2x+1) - \frac{3m^2(m^2-7)}{2}\right)^2, \tag{4.14}$$

and

$$\widetilde{z} := 144\widetilde{m}^2 - \left(3\widetilde{v} + \left(8y + 4x^3 + 6x^2 + 3\right)(2x+1) - \frac{3\widetilde{m}^2(\widetilde{m}^2-7)}{2}\right)^2. \tag{4.15}$$

where

$$\widetilde{v} := \widetilde{n} + 1 - \frac{y\widetilde{n} + \mu - xy}{x} \quad \text{and} \quad \widetilde{n} := \widetilde{m}^2 + 4x^2 + 4x - 2. \tag{4.16}$$

We substitute Eqs. (4.4), (4.11) and (4.13) into Eq. (4.15) and set

$$G_5(a,b,c;x) := \widetilde{z} \in \mathbb{Q}[a,b,c][x^{-1}].$$

Thus, we can express G_5 as

$$G_5(a,b,c;x) = \sum_{i=-20}^{-2} G_{5,i} x^i,$$

for some $G_{5,i} \in \mathbb{Q}[a,b,c]$. Use computers do the following things:

Step 5(a) Give a good upper bound on $G_{5,i}$ when $a \in [-1,3]$, $b \in [-3994,64]$ and $c \in [-1620,3]$.

Step 5(b) Use **Step 5**(a) to prove that $|G_5(a,b,c;x)| < 1$ if $x \geq 120$, $a \in [-1,3]$, $b \in [-3994,64]$ and $c \in [-1620,3]$.

In this step, let (x_0, y_0, μ_0) be a tuple in D such that x_0, y_0, μ_0, n_0, v_0 are all integers, $x_0 \geq 120$ and $F_3(x_0,y_0,\mu_0) = 0$. By Eq. (4.14), we know that z_0 is an integer.

Consider an additional equation $m_0^2 = \widetilde{m}_0^2$, which implies that $\widetilde{n}_0 = n_0$ by Eqs. (4.1) and (4.16), $\widetilde{v}_0 = v_0$ by Eqs. (4.2) and (4.16) and $\widetilde{z}_0 = z_0$ by Eqs. (4.14) and (4.15). **Step 4** shows that Eq. (4.4) defines an $a_0 \in [-1,3]$, Eq. (4.11) defines a $b_0 \in [-3994,64]$ and the equation $m_0^2 = \widetilde{m}_0^2$ defines a $c_0 \in [-1620,3]$. Then, **Step 5**(a) and **Step 5**(b) prove that $|\widetilde{z}_0| < 1$.

Since $z_0 = \widetilde{z}_0$, we know that z_0 is an integer and $|z_0| < 1$. Therefore, $z_0 = 0$. According to Eq. (4.14), $144m_0^2 = 144m_0^2 - z_0$ is the square of an integer. Thus, m_0 is an integer. Since Re $m_0 \geq 0$, m_0 is a nonnegative integer. **Step 4** shows that $m_0^2 < 9$, then we have $m_0 \in \{0,1,2\}$.

Step 6 In **Step 6** and **Step 7**, we assume that $m = m_0$ for some $m_0 \in \{0,1,2\}$, and the goal of these two steps is to find out suitable solutions of the system of equations $F_3(x,y,\mu) = 0$ and $m = m_0$.

Consider the expression

$$\mu(y-x)(m^2 - m_0^2). \tag{4.17}$$

We substitute Eqs. (4.1) and (4.12) into Eq. (4.17) and set

$$G_6(m_0;x,y;\mu) := \mu(y-x)(m^2 - m_0^2) \in \mathbb{Z}[m_0][x,y][\mu].$$

Regarding polynomials $G_6(m_0; x, y, \mu) \in \mathbb{Z}[m_0][x, y][\mu]$ and $F_3(x, y, \mu) \in \mathbb{Z}[x, y][\mu]$ as polynomials in a single variable μ, we apply the extended Euclidean algorithm to them and get

$$F_3(x, y, \mu)p(m_0; x, y) + G_6(m_0; x, y; \mu)q(m_0; x, y) = x^2(x+1)y^2(y+1)^3(y-x)^2F_4(m_0; x, y) \quad (4.18)$$

for some nonzero minimal polynomials $p(m_0; x, y), q(m_0; x, y), F_4(m_0; x, y) \in \mathbb{Z}[m_0][x, y]$. Moreover, the choice for $F_4(m_0; x, y)$ is unique up to sign. We use the convention that $F_4(m_0; x, y)$ is the unique nonzero minimal polynomial satisfying Eq. (4.18) such that the coefficient of x^{13} in $F_4(m_0; x, y)$ is positive.

Step 7 In this step, we find all large solutions of $F_4(m_0; x, y) = 0$ in a certain region, for $m_0 \in \{0, 1, 2\}$. The polynomial $F_4(m_0; x, y)$ is of degree 3 in y. Let

$$\begin{aligned} y^{(1)} &:= -\left(2x^3 + 3x^2 + \frac{3m_0(m_0+1)}{2}x + \frac{3m_0(m_0+1)}{4}\right), \\ y^{(2)} &:= -\left(2x^3 + 3x^2 + \frac{3m_0(m_0-1)}{2}x + \frac{3m_0(m_0-1)}{4}\right), \\ y^{(3)} &:= x. \end{aligned}$$

Note that when $x \geq 3$ and $m_0 = 0$, we have

$$y^{(1)} < y^{(2)} + \frac{1}{x} < y^{(2)} + \frac{1}{2} < y^{(3)} - 1 < y^{(3)} < y^{(3)} + 1 \quad (4.19)$$

and when $x \geq 1$ and $m_0 \in \{1, 2\}$, we have

$$y^{(1)} - \frac{1}{2} < y^{(1)} < y^{(1)} + \frac{1}{2} < y^{(2)} - \frac{1}{2} < y^{(2)} < y^{(2)} + \frac{1}{2} < y^{(3)} - 1 < y^{(3)} < y^{(3)} + 1. \quad (4.20)$$

Use computers to prove that:

Step 7(a) $F_4(m_0; x, y^{(1)}) = 0$ if $m_0 = 0$.

Step 7(b) $F_4(m_0; x, y^{(1)} - \frac{1}{2}) > 0$ if $x \geq 90$ and $m_0 \in [1, 2]$.

Step 7(c) $F_4(m_0; x, y^{(1)} + \frac{1}{2}) < 0$ if $x \geq 90$ and $m_0 \in [1, 2]$.

Step 7(d) $F_4(m_0; x, y^{(2)} + \frac{1}{x}) < 0$ if $x \geq 90$ and $m_0 = 0$.

Step 7(e) $F_4(m_0; x, y^{(2)} - \frac{1}{2}) < 0$ if $x \geq 90$ and $m_0 \in [1, 2]$.

Step 7(f) $F_4(m_0; x, y^{(2)} + \frac{1}{2}) > 0$ if $x \geq 90$ and $m_0 \in [0, 2]$.

Step 7(g) $F_4(m_0; x, y^{(3)} - 1) > 0$ if $x \geq 1$ and $m_0 \in [0, 2]$.

Step 7(h) $F_4(m_0; x, y^{(3)} + 1) < 0$ if $x \geq 1$ and $m_0 \in [0, 2]$.

Let $x_0 \geq 90$ be an integer. Consider the polynomial $F_4(m_0; x_0, y)$ in single variable y of degree 3.

When $m_0 = 0$, **Step 7**(a) gives a solution $y_0^{(1)} \in (y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2})$, **Step 7**(d) and **Step 7**(f) gives a solution in $(y_0^{(2)} + \frac{1}{x}, y_0^{(2)} + \frac{1}{2}) \subset (y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2})$, **Step 7**(g) and **Step 7**(h) gives a solution in $(y_0^{(3)} - 1, y_0^{(3)} + 1)$, and these three solutions are all different by Eq. (4.19). Therefore, when $m_0 = 0$, all solutions of $F_4(m_0; x_0, y)$ are in

$$(y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2}) \cup (y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2}) \cup (y_0^{(3)} - 1, y_0^{(3)} + 1). \quad (4.21)$$

When $m_0 \in \{1, 2\}$, **Step 7**(b) and **Step 7**(c) give a solution in $(y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2})$, **Step 7**(e) and **Step 7**(f) give a solution in $(y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2})$, **Step 7**(g) and **Step 7**(h) give a solution in $(y_0^{(3)} - 1, y_0^{(3)} + 1)$, and these three solutions are all different by Eq. (4.20). Therefore, when $m_0 \in \{1, 2\}$, all solutions of $F_4(m_0; x_0, y)$ are also in Eq. (4.21).

Now, let $y_0 \leq -1$ be an integer such that $F_4(m_0; x_0, y_0) = 0$. By the discussion above, we know that y_0 is in

$$(y_0^{(1)} - \frac{1}{2}, y_0^{(1)} + \frac{1}{2}) \cup (y_0^{(2)} - \frac{1}{2}, y_0^{(2)} + \frac{1}{2}) \cup (y_0^{(3)} - 1, y_0^{(3)} + 1). \tag{4.22}$$

Since $m_0 \in \{0, 1, 2\}$, all of $y_0^{(1)}$, $y_0^{(2)}$ and $y_0^{(3)}$ are in $\frac{1}{2}\mathbb{Z}$. Therefore, Eq. (4.22) contains a unique negative integer $-x_0^2(2x_0 + 3)$ when $m_0 \in \{0, 1\}$ and no integers when $m_0 = 2$.

Therefore, for $m_0 \in \{0, 1, 2\}$, all integer solutions (x_0, y_0) of $F_4(m_0; x, y) = 0$ such that $x_0 \geq 90$ and $y_0 \leq -1$ are $(t_0, -t_0^2(2t_0 + 3))$ for positive integer t_0.

Step 8 Use computers to prove that, for all integer $x_0 \in [1, 120]$ and all negative integer $y_0 \in [-(2x_0^3 + 3x_0^2 + 3x_0 + 2), -(2x_0^3 + 3x_0^2 - 3x_0 - 3] \setminus \{-x_0^2(2x_0 + 3)\}$, the polynomial $F_3(x_0, y_0, \mu)$ in μ has a positive integer solution if and only if $x_0 = 1$ and $y_0 = -1$. Moreover, $F_3(1, -1, \mu)$ has a unique positive integer solution $\mu_0 = 1$.

Step 9 It is easy to check all tuples in Table 4.1 are solutions of $F_3(x, y, \mu)$.

Let (x_0, y_0, μ_0) be a tuple in D such that n_0 and v_0 are integers and $F_3(x_0, y_0, \mu_0) = 0$.

By Eq. (4.3) in **Step 1**, $y_0 \in [-(2x_0^3 + 3x_0^2 + 3x_0 + 2), -(2x_0^3 + 3x_0^2 - 3x_0 - 3)]$. Then, by **Step 8**, either $x_0 \geq 120$, or $(x_0, y_0, \mu_0) = (1, -1, 1)$ which is in Table 4.1, or $y_0 = -x_0^2(2x_0 + 3)$. For the last case, there are exactly three solutions of $F_3(x_0, y_0, \mu) = 0$ in μ: $-x_0 y_0$ which is in Table 4.1, $-(x_0 + 1)y_0$ which is also in Table 4.1, and $-2x_0^4 - x_0^3 + x_0^2$ which is negative. Therefore, Table 4.1 contains all solutions when $x_0 \leq 120$. Now, assume that $x_0 \geq 120$.

By **Step 5**, $m_0 \in \{0, 1, 2\}$. Then, **Step 6** shows that $F_4(m_0; x_0, y_0) = 0$. **Step 7** solves $F_4(m_0; x_0, y_0) = 0$ and gives $y_0 = -x_0^2(2x_0 + 3)$. By the discussion above, all solutions with $y_0 = -x_0^2(2x_0 + 3)$ are in Table 4.1.

Therefore, all solutions are found and the proof of Theorem 4.1 is finished. □

Remarks from Nozaki

After we finish the manuscript, we receive a remark from Hiroshi Nozaki who suggests us that there exists a short approach to Theorem 1.1. Let X be a spherical 2-distance $\{4, 2, 1\}$-design on S^{n-1}. If X is a $\{3\}$-design, then X is a spherical 2-distance 4-design, hence by [13] it is a tight spherical 4-design. If X is not a spherical $\{3\}$-design, then X is a spherical 2-distance 2-design but not a 3-design. Therefore, in the latter case, by [15] or [16], we have $|X| \leq \frac{n(n+1)}{2}$. Consider the multiset $X' := X \cup (-X)$, then $|X'| \leq n(n + 1)$. Since X' is antipodal, X' is a $\{5, 3\}$-design with possibly repeated points. In conjunction with $\{4, 2, 1\}$-design assumption, X' is a spherical 5-design, which has the lower bound $|X'| \geq n(n + 1)$ by [13]. Therefore, X' is a tight spherical 5-design.

Although above discussion covers Theorem 1.1, we want to emphasize that our method is important since there are a lot of classification problems which can be reduced to the problems of solving some Diophantine equations. The prototype of our method had been used to prove the nonexistence of nontrivial tight combinatorial 8-design [19]. In an ongoing project, we have obtained some partial results on complex spherical 4-distance T-design by solving Diophantine equations. Our ultimate goal is to get the classification of spherical s-distance $(2s - 1)$-designs for $s \geq 3$ by this method.

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