A Fisher type inequality for weighted regular t-wise balanced designs

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Abstract

Delsarte and Seidel found that the dimension of a certain linear space can serve as a Fisher type lower bound for the size of a weighted regular t-wise balanced design. They put it as a problem to explicitly determine this dimension. This note calculates the rank of a special set inclusion matrix and then solves the above problem.

1 Introduction

All arithmetic computations in this note are carried out over the field of reals.

For any nonnegative integer n, we use the notation $[n]$ for the set $\{0, 1, \ldots, n\}$. We fix a positive integer v. For any positive integer t, a *weighted regular t-wise balanced design* with *point* set $[v]$ is a pair $(\mathcal{B}, w)$, where $\mathcal{B} \subseteq 2^{[v]}$ is the set of *blocks* and $w \in \mathbb{R}_+^{\mathcal{B}}$ is a strictly positive *weight function* over $\mathcal{B}$ such that for any $S \in \binom{[v]}{\leq t}$ the number $\sum_{\substack{L \in \mathcal{B} \\ S \subseteq L}} w(L)$ is determined by $|S|$. Note that Delsarte and Seidel [1, §6] refer to a weighted regular t-wise balanced design as a Boolean design with indices from $[t]$.

For any set systems $\mathcal{H}$ and $\mathcal{G}$, the *inclusion matrix* of $\mathcal{H}$ vs $\mathcal{G}$ is the $|\mathcal{H}|$ by $|\mathcal{G}|$ matrix $M \in \mathbb{R}^{\mathcal{H} \times \mathcal{G}}$ satisfying

$$M(H, G) = \begin{cases} 1, & \text{if } H \subseteq G, \\ 0, & \text{otherwise,} \end{cases}$$

for each $H \in \mathcal{H}$ and $G \in \mathcal{G}$. For any two nonnegative integers a and b we write $\mathbf{M}_{a,b}$ for the inclusion matrix of $\binom{[v]}{a}$ vs $\binom{[v]}{b}$. For any $A, B \subseteq [v]$, let $\mathbf{M}_{A,B}$ denote the $(\cup_{a \in A} \binom{[v]}{a}) \times (\cup_{b \in B} \binom{[v]}{b})$ matrix such that $\mathbf{M}_{A,B}(\binom{[v]}{a}, \binom{[v]}{b}) = \mathbf{M}_{a,b}$ for every $a \in A$ and $b \in B$.

Delsarte and Seidel obtain the following Fisher type lower bound for the size of a weighted regular t-wise balanced design.

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Theorem 1 (Delsarte-Seidel [1, Theorem6.3]). *Let $t = 2e$ be an even positive integer. Suppose $(\mathcal{B}, w)$ is a weighted regular t-wise balanced design over $[v]$ and B is the set of block sizes of $\mathcal{B}$. If $e \le b$ for every $b \in B$, then*

$$|\mathcal{B}| \ge \operatorname{rank} \mathbf{M}_{[e],B}.$$

In view of the above theorem, it is of interest to determine $\operatorname{rank} \mathbf{M}_{[e],B}$. Delsarte and Seidel mention this as an open problem [1, p. 228] and report some partial results [1, p. 229]. In this note we will prove the following theorem:

Theorem 2. *Let $A = \{a_0, \ldots, a_p\}$ and $B = \{b_0, \ldots, b_q\}$ be subsets of $[v]$ where $a_p < \cdots < a_0 \le b_0 < \cdots < b_q$. Then*

$$\operatorname{rank} \mathbf{M}_{A,B} = \sum_{i \in [\min\{p,q\}]} \min\left\{\binom{v}{a_i}, \binom{v}{b_i}\right\}. \tag{1}$$

By taking $A = [e]$ in Theorem 2 , we arrive at the next result, which settles the problem of Delsarte and Seidel.

Theorem 3. *Let e be a positive integer and let $B = \{b_0, \ldots, b_q\}$ with $e \le b_0 < b_1 < \cdots < b_q$. Then we have*

$$\operatorname{rank} \mathbf{M}_{[e],B} = \sum_{i \in [q]} \min\left\{\binom{v}{e-i}, \binom{v}{b_i}\right\}.$$

The following is an immediate consequence of Theorems 1 and 3.

Corollary 4. *Let e be a positive integer and let $(\mathcal{B}, w)$ be a weighted regular $2e$-wise balanced design over $[v]$. If B is the set of block sizes of $(\mathcal{B}, w)$ and $e \le b \le v - e$ for every $b \in B$, then*

$$|\mathcal{B}| \ge \operatorname{rank} \mathbf{M}_{[e],B} = \sum_{i \in [|B|-1]} \binom{v}{e-i}.$$

2 Proof of Theorem 2

When we consider a matrix whose row index set and column index set are sets of integers, we adopt the convention that the rows/columns are arranged so that the ones with smaller indices come earlier. For example, if we say that a matrix $U \in \mathbb{R}^{[p]\times[p]}$ is lower triangular, this means that $U(i, j) = 0$ whenever $i < j$.

A matrix is *strictly totally positive* if all its minors are positive [6]. When we fix an ordering of the row index set and an ordering of the column index set, a *successive minor* of a matrix is the determinant of a submatrix whose row indices and column indices are both consecutive in the given orderings.

Theorem 5 (Fekete's Lemma [6, Lemma 2.1]). *A matrix is strictly totally positive if and only if all its successive minors are positive.*

Lemma 6 ([5, Lemma 3])**.** *Let $X_0, \ldots, X_n$, $Y_1, \ldots, Y_n$, and $Z_1, \ldots, Z_n$ be indeterminates. Then*

$$\det_{i,j\in[n]} \left(\prod_{j<k\le n} (X_i+Y_k) \prod_{1\le k\le j} (X_i+Z_k) \right) = \prod_{0\le i<j\le n} (X_i - X_j) \prod_{1\le i\le j\le n} (Z_i - Y_j).$$

For any two positive integers $m \le n$, let $\mathbf{A}_{m|n}$ be the $(m+1)\times(m+1)$ matrix satisfying $\mathbf{A}_{m|n}(i,j) = \frac{1}{(n+i-j)!}$ for $i, j \in [m]$.

Corollary 7. $\det \mathbf{A}_{a|b} = \prod_{i\in[a]} \frac{i!}{(b+i)!} > 0$.

Proof. Taking all the Y_i to infinity, it is a direct consequence of Lemma 6 that

$$\det_{i,j\in[a]} \left(\prod_{1\le k\le j} (X_i + Z_k) \right) = \prod_{0\le i<j\le a} (X_j - X_i).$$

Letting $X_i = b+i$ for $i \in [a]$ and $Z_k = -k$ for $k = 1, \ldots, a$ we get

$$\det_{i,j\in[a]} \left(\prod_{1\le k\le j} (b+i-k) \right) = \prod_{i\in[a]} i!. \tag{2}$$

Finally, we have

$$\begin{aligned} \det \mathbf{A}_{a|b} &= \det_{i,j\in[a]} \left(\frac{1}{(b+i-j)!} \right) \\ &= \left(\prod_{i\in[a]} \frac{1}{(b+i)!} \right) \det_{i,j\in[a]} \left(\prod_{1\le k\le j} (b+i-k) \right) \\ &= \prod_{i\in[a]} \frac{i!}{(b+i)!}, \quad \text{(by Eq.(2))} \end{aligned}$$

as was to be shown. □

Corollary 8. *For any positive integer n, the matrix $\mathbf{A}_{n|n}$ is strictly totally positive.*

Proof. Take an arbitrary successive minor of $\mathbf{A}_{n|n}$, which must have the form $\det \mathbf{A}_{a|b}$ for some integers a and b satisfying $0 \le a \le b \le 2n-a$. It is now clear that the result follows from Theorem 5 and Corollary 7, as desired. □

The following results are well-known.

Lemma 9 (Kantor [4, p. 315])**.** *For any $a, b \in [v]$,* $\operatorname{rank} \mathbf{M}_{a,b} = \min\{\binom{v}{a}, \binom{v}{b}\}$.

Lemma 10 (Kantor [4, p. 317])**.** *If $0 \le a \le b \le c \le v$, then*

$$\frac{(c-a)!}{(c-b)!(b-a)!} \mathbf{M}_{a,c} = \mathbf{M}_{a,b} \mathbf{M}_{b,c}.$$

For any two nonnegative integers a and b we set $\mathbf{N}_{a,b} = (b-a)!\mathbf{M}_{a,b}$. For any $a \le b \le c$, it follows from Lemma 10 that

$$\mathbf{N}_{a,b}\mathbf{N}_{b,c} = \mathbf{N}_{a,c}. \tag{3}$$

Let us fix two subsets $A = \{a_0, \dots, a_p\}$ and $B = \{b_0, \dots, b_q\}$ of $[v]$ satisfying

$$a_p < \cdots < a_0 \le b_0 < \cdots < b_q. \tag{4}$$

For any matrix $U \in \mathbb{R}^{[p]\times[p]}$, let $\psi(U) \in \mathbb{R}^{(\cup_{a\in A}\binom{[v]}{a})\times(\cup_{a\in A}\binom{[v]}{a})}$ be specified by

$$\psi(U)\Big(\binom{[v]}{a_i}, \binom{[v]}{a_\ell}\Big) = U(i,\ell)\mathbf{N}_{a_i,a_\ell}, \forall i \in [p], \ell \in [p];$$

for any matrix $K \in \mathbb{R}^{[p]\times[q]}$, let $\phi(K) \in \mathbb{R}^{(\cup_{a\in A}\binom{[v]}{a})\times(\cup_{b\in B}\binom{[v]}{b})}$ be specified by

$$\phi(K)\Big(\binom{[v]}{a_\ell}, \binom{[v]}{b_m}\Big) = K(\ell,m)\mathbf{N}_{a_\ell,b_m}, \forall \ell \in [p], m \in [q];$$

for any matrix $V \in \mathbb{R}^{[q]\times[q]}$, let $\tau(V) \in \mathbb{R}^{(\cup_{b\in B}\binom{[v]}{b})\times(\cup_{b\in B}\binom{[v]}{b})}$ be specified by

$$\tau(V)\Big(\binom{[v]}{b_m}, \binom{[v]}{b_j}\Big) = V(m,j)\mathbf{N}_{b_m,b_j}, \forall m \in [q], j \in [q].$$

An important consequence of Eq. (3) is

$$\psi(U)\phi(K)\tau(V) = \phi(UKV) \tag{5}$$

for every lower triangular matrix $U \in \mathbb{R}^{[p]\times[p]}$, every $K \in \mathbb{R}^{[p]\times[q]}$ and every upper triangular matrix $V \in \mathbb{R}^{[q]\times[q]}$.

Proof of Theorem 2. Let $\mathbf{K} \in \mathbb{R}^{[p]\times[q]}$ be the matrix with $\mathbf{K}(i,j) = \frac{1}{(b_j-a_i)!}$ for every $i \in [p]$ and $j \in [q]$. It is not hard to check that

$$\phi(\mathbf{K}) = \mathbf{M}_{A,B}. \tag{6}$$

Moreover, it follows from Corollary 8 that $\mathbf{K}$ is a strictly totally positive matrix. This allows us to apply Gaussian elimination and obtain a unique LDU factorization of $\mathbf{K}$ ([2, §3.1],[3, pp. 10–11], [6, p. 51]), that is,

$$\mathbf{K} = UKV, \tag{7}$$

where U is a unit lower triangular matrix, K a diagonal matrix and V a unit upper triangular matrix. Indeed, the three matrices $U \in \mathbb{R}^{[p]\times[p]}$, $K \in \mathbb{R}^{[p]\times[q]}$

and $V \in \mathbb{R}^{[q]\times[q]}$ are explicitly given by

$$U(i,j) = \begin{cases} \frac{\det \mathbf{K}([j-1]\cup\{i\},[j])}{\det \mathbf{K}([j],[j])}, & j \in [\min\{p,q\}], i \geq j, \\ 0, & j \in [\min\{p,q\}], i < j, \\ \delta_{i,j}, & j > \min\{p,q\}, \end{cases}$$

$$K(i,j) = \begin{cases} \frac{\det \mathbf{K}([i],[i])}{\det \mathbf{K}([i-1],[i-1])}, & i = j \in [\min\{p,q\}], \\ 0, & \text{else}, \end{cases}$$

$$V(i,j) = \begin{cases} \frac{\det \mathbf{K}([i],[i-1]\cup\{j\})}{\det \mathbf{K}([i],[i])}, & i \in [\min\{p,q\}], j \geq i, \\ 0, & j \in [\min\{p,q\}], j < i, \\ \delta_{i,j}, & i > \min\{p,q\}, \end{cases}$$

where we have used δ for the Kronecker delta function and have adopted the convention that $\det \mathbf{K}(\emptyset,\emptyset) = 1$.

Note that both $\psi(U)$ and $\tau(V)$ are nonsingular matrices. Accordingly, Eqs. (5), (6) and (7) tell us that $\operatorname{rank} \mathbf{M}_{A,B} = \operatorname{rank} \phi(K)$. By virtue of Lemma 9, the RHS of Eq. (1) is just $\operatorname{rank} \phi(K)$ and this ends the proof. □

The main work in this note is to get a formula for $\operatorname{rank} \mathbf{M}_{A,B}$ on the condition that Eq. (4) is satisfied. A natural question is to compute $\operatorname{rank} \mathbf{M}_{A,B}$ when Eq. (4) does not necessarily hold. In this general case, we can still treat each block $\mathbf{M}_{a_i,b_j}$ as an individual element and then carry out block Gaussian elimination to find that $\operatorname{rank} \mathbf{M}_{A,B} = \sum \operatorname{rank} \mathbf{M}_{a_i,b_j}$ where (i,j) runs through a subset S of $[p] \times [q]$ with no two elements sharing the same first or second coordinate. It would be interesting to determine this set S in the general case.

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