An explicit construction of spherical designs

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Definition 1

A finite subset $X \subseteq S^d$ is a **spherical $t$-design** provided that

$$
\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{\nu^d(S^d)} \int_{S^d} f \, d\nu^d
$$

for all $f \in \mathbb{R}[x_0, \ldots, x_d]_{\leq t}$, where $\nu^d$ is the spherical measure on $S^d$. 
Spherical designs

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Related concept:

- Weighted design ($X = (X, \mu_X)$).
- Rational design ($X \subseteq \mathbb{Q}^{d+1}$).
- Semidesign ($f \in \mathbb{R}[x_1, \ldots, x_d]_{\leq t}$).
- Rational-weighted rational semidesign.
Constructions of designs

- **Definition**: Delsarte-Goethals-Seidel (1977)
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Problem 2: Are there rational spherical $t$-designs on $S^d$ for all large $d$?
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**Problem 2**

Are there **rational** spherical $t$-designs on $S^d$ for all large $d$?
Structure of sphere and hemisphere as topological space

Let

\[ H^d := \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} : x_0 > 0\} \]

be the \(d\)-dimensional open hemisphere.
Structure of sphere and hemisphere as topological space

Let

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be the \(d\)-dimensional open hemisphere.

There exists a dominant open embedding of topological spaces

\[ S^a \times H^b \rightarrow S^{a+b} \]

\[ (x_0, \ldots, x_a) \times (y_0, \ldots, y_b) \mapsto (x_0 y_0, \ldots, x_a y_0, y_1 \ldots, y_b). \]
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Structure of sphere and hemisphere as measure space

Let $\mathcal{H}^d_s := (H^d, \nu^d_s)$ for certain measure $\nu^d_s$ on $H^d$. (The Radon-Nikodym derivative of $\nu^d_s$ with respect to the spherical measure $\nu^d$ is the polynomial $x_0 \mapsto x^s_0$.)
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There exists a dominant open embedding of measure spaces

$$S^a \times \mathcal{H}_a^b \to S^{a+b},$$

and an isomorphism of measure spaces

$$\mathcal{H}_s^a \times \mathcal{H}_a^{b+s} \to \mathcal{H}_s^{a+b}.$$
Structure of sphere and hemisphere as measure space

Let $\mathcal{H}_s^d := (H^d, \nu_s^d)$ for certain measure $\nu_s^d$ on $H^d$. (The Radon-Nikodym derivative of $\nu_s^d$ with respect to the spherical measure $\nu^d$ is the polynomial $x_0 \mapsto x_0^s$.)

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$$S^a \times \mathcal{H}_a^b \rightarrow S^{a+b},$$

and an isomorphism of measure spaces

$$\mathcal{H}_s^a \times \mathcal{H}_{a+s}^b \rightarrow \mathcal{H}_{s}^{a+b}.$$

**Proposition 3**

*There exists a dominant open embedding of measure spaces*

$$S^1 \times (\mathcal{H}_1^1 \times \cdots \times \mathcal{H}_{d-1}^1) \rightarrow S^d.$$
Sketch of an explicit construction of spherical designs

ERRS: explicit rational-weighted rational semidesign.

1. ERRS on $\mathcal{H}_0^1$.
2. ERRS on $\mathcal{H}_1^1$.
3. ERRS on $\mathcal{H}_s^1$.
4. ERRS on $\mathcal{H}_1^{d-1} \cong \mathcal{H}_1^1 \times \cdots \times \mathcal{H}_{d-1}^1$.
5. Explicit integer-weighted rational semidesign on $\mathcal{H}_1^{d-1}$.
6. Explicit design on $S^1$.
7. Explicit design on $S^d \sim S^1 \times \mathcal{H}_1^{d-1}$. 
Step 1. Rational-weighted rational semidesign on $\mathcal{H}_0^1$

**Theorem 4**

Choose $(b_i, a_i)$ in $H^1 \cap \mathbb{Q}^2$ such that

$$\left| a_i - \sin \left( \frac{-t + 2i + 1)\pi}{2t} \right) \right| < \frac{\pi 2^t}{2^t t^{2t}}.$$

Then, $X := \{(b_i, a_i)\}$ is the support of a unique rational-weighted rational $(t - 1)$-semidesign $X^1_0 = (X, \mu^1_0)$ on $\mathcal{H}_0^1$. Moreover,

$$\mu^1_0(b_i, a_i) = \sum_{\text{even } j=0}^{t-1} \frac{e_{t-j-1}(a_1, \ldots, \hat{a}_i, \ldots, a_t)}{(j + 1) \prod_{k \in [0, t-1] \setminus i} (a_k - a_i)},$$

where $e_{t-j-1}$ is the $(t - j - 1)$-th elementary symmetric polynomial.
Step 2. Rational-weighted rational semidesign on $\mathcal{H}_1^1$

**Theorem 5**

Assume that $n$ is an odd integer multiple of even integer $t$ and $n > t^{t/2}$. Choose $(b_i, a_i)$ in $H^1 \cap \mathbb{Q}^2$ such that

$$\left| a_i - \frac{-n + 1 + 2i}{n} \right| < \frac{t}{2n^4}.$$

Let $(b'_i, a'_i) := (b_j, a_j)$ where $j = \frac{(2i+1)n-t}{t}$. Then, $X := \{(b_i, a_i)\}$ is the support of a unique rational-weighted rational $(t - 1)$-semidesign $\mathcal{X}_0^1 = (X, \mu_0^1)$ on $\mathcal{H}_0^1$ such that $\mu_0^1(b_i, a_i) = 1$ for $(b_i, a_i) \notin \{(b'_i, a'_i)\}$. Moreover,

$$\mu_0^1(b'_i, a'_i) = 1 + \sum_{j=0}^{t-1} (-1)^j \frac{e_{t-j-1}(a'_1, \ldots, \hat{a}'_i, \ldots, a'_t)}{\prod_{k \in [0, t-1]} (a'_k - a'_i)} \epsilon_{n,j}$$

where $\epsilon_{n,j} := \frac{1}{n} \sum_{i=0}^{n-1} a'_i - \frac{1 + (-1)^j}{2(j+1)}$. 
Step 3. Rational-weighted rational semidesign on $\mathcal{H}_s^1$

**Lemma 6**

Let $\mathcal{X}_s^d = (X, \mu_s^d)$ be a rational-weighted rational $(t + \tilde{s} - s)$-semidesign on $\mathcal{H}_s^d$, where $\tilde{s} - s$ is nonnegative even. Then, $\mathcal{X}_{s\rightarrow \tilde{s}}^d := (X, \mu_{s\rightarrow \tilde{s}}^d)$ is a rational-weighted rational $t$-semidesign on $\mathcal{H}_{\tilde{s}}^d$, where

$$\mu_{s\rightarrow \tilde{s}}^d(x_0, \ldots, x_d) := x_0^{\tilde{s} - s} \mu_s^d(x_0, \ldots, x_d).$$
Step 3. Rational-weighted rational semidesign on $\mathcal{H}_s^1$

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$$\mu_{s \rightarrow \tilde{s}}^d(x_0, \ldots, x_d) := x_0^{\tilde{s} - s} \mu_s^d(x_0, \ldots, x_d).$$

**Corollary 7**

Let $\mathcal{X}_0^1$ be a rational-weighted rational $(t + s)$-semidesign on $\mathcal{H}_0^1$ and $\mathcal{X}_1^1$ a rational-weighted rational $(t + s - 1)$-semidesign on $\mathcal{H}_1^1$. Then, $\mathcal{X}_{i \mod 2 \rightarrow i}^1$ is a rational-weighted rational $t$-semidesign on $\mathcal{H}_s^1$. 
Step 4. Rational-weighted rational semidesign on $\mathcal{H}_1^{d-1}$

Lemma 8

Let $\mathcal{X}_0$ be a rational-weighted design on $\mathcal{Z}_0$ and $\mathcal{X}_1$ a rational-weighted design on $\mathcal{Z}_1$. Then, $\mathcal{X}_0 \times \mathcal{X}_1$ is a rational-weighted design on $\mathcal{Z}_0 \times \mathcal{Z}_1$. 
Step 4. Rational-weighted rational semidesign on $\mathcal{H}_1^{d-1}$

**Lemma 8**

Let $\mathcal{X}_0$ be a rational-weighted design on $\mathbb{Z}_0$ and $\mathcal{X}_1$ a rational-weighted design on $\mathbb{Z}_1$. Then, $\mathcal{X}_0 \times \mathcal{X}_1$ is a rational-weighted design on $\mathbb{Z}_0 \times \mathbb{Z}_1$.

**Corollary 9**

For each $s \in [1, d-1] \mathbb{Z}$, let $\mathcal{X}_s^1$ be a rational-weighted rational $t$-semidesign. Then,

$$\mathcal{X}_1^{d-1} := \mathcal{X}_1^1 \times \cdots \times \mathcal{X}_1^{d-1}$$

is a rational-weighted rational $t$-semidesign on $\mathcal{H}_1^{d-1} \cong \mathcal{H}_1^1 \times \cdots \times \mathcal{H}_1^{d-1}$. 
Lemma 10

Let $\mathcal{X} = (X, \mu_X)$ be a rational-weighted design on $\mathbb{Z}$. Then, $\overline{\mathcal{X}} := (X, n_X \mu_X)$ is an integer-weighted design on $\mathbb{Z}$, where

$$n_X := \text{lcm}_{x \in X} \text{ denominator of } \mu_X(x).$$
Step 5. Integer-weighted rational semidesign on $\mathcal{H}_1^{d-1}$

Lemma 10

Let $\mathcal{X} = (X, \mu_X)$ be a rational-weighted design on $\mathbb{Z}$. Then, $\overline{\mathcal{X}} := (X, n_X \mu_X)$ is an integer-weighted design on $\mathbb{Z}$, where

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Corollary 11

Let $\mathcal{X}_1^{d-1}$ be a rational-weighted rational $t$-semidesign on $\mathcal{H}_1^{d-1}$. Then, $\overline{\mathcal{X}}_1^{d-1}$ is an integer-weighted rational $t$-semidesign on $\mathcal{H}_1^{d-1}$. 
Step 6. Designs on $S^1$

Proposition 12

Let $X$ be the vertices of a regular $(t + 1)$-gon in $S^1$. Then, $X$ is a $t$-design on $S^1$. 
Lemma 13

Let \( \mathcal{X}_0 \) be a design on \( \mathbb{Z}_0 \) and \( \mathcal{X}_1 \) an integer-weighted design on \( \mathbb{Z}_1 \). Let \( g : (0, 1) \rightarrow \text{Aut}(\mathbb{Z}_0) \) be a map such that \( g(s) \mathcal{X}_0 \cap g(s') \mathcal{X}_0 = \emptyset \) for different \( s, s' \in (0, 1) \). Then,

\[
\mathcal{X}_0 \bowtie \mathcal{X}_1 := \{ (g(s_{x_1,i})x_0, x_1) : x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1, i \in [1, \mu_{\mathcal{X}_1}(x_1)]_\mathbb{Z} \}
\]

is a design on \( \mathbb{Z}_0 \times \mathbb{Z}_1 \), provided that \( s_{x_1,i} \)'s are distinct numbers in \( (0, 1) \).
Step 7. Designs on $S^d$

**Lemma 13**

Let $\mathcal{X}_0$ be a design on $\mathbb{Z}_0$ and $\mathcal{X}_1$ an integer-weighted design on $\mathbb{Z}_1$. Let $g : (0, 1) \to \text{Aut}(\mathbb{Z}_0)$ be a map such that $g(s) \mathcal{X}_0 \cap g(s') \mathcal{X}_0 = \emptyset$ for different $s, s' \in (0, 1)$. Then,

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is a design on $\mathbb{Z}_0 \times \mathbb{Z}_1$, provided that $s_{x_1,i}$'s are distinct numbers in $(0, 1)$.

**Corollary 14**

Let $\mathcal{Y}_1$ be a design on $S^1$ and $\mathcal{X}_1^{d-1}$ an integer-weighted $t$-semidesign. Then,

$$\mathcal{Y}_1 \bowtie \mathcal{X}_1^{d-1}$$

is a design on $S^d$. 

\textbf{Theorem 15}

Let $\mathcal{Y}^1$ be an explicit $t$-design on $S^1$, $\mathcal{X}_0^1$ an explicit rational-weighted rational $(t + d - 2)$-semidesign on $\mathcal{H}_0^1$ and $\mathcal{X}_1^1$ an explicit rational-weighted rational $(t + d - 1)$-semidesign on $\mathcal{H}_1^1$. Then,

$$\mathcal{Y}^1 \rtimes \prod_{i=1}^{d-1} \mathcal{X}_{i \mod 2 \rightarrow i}^1$$

is an explicit spherical $t$-design on $S^d$. 

\textbf{Remark 16}

Designs above can be constructed over $\mathbb{Q}^{ab} \cap \mathbb{Q}$. Designs of arbitrary large size can be constructed.

Thank you for your attention.
Explicit spherical design

Theorem 15

Let $\mathcal{Y}^1$ be an explicit $t$-design on $S^1$, $\mathcal{X}^1_0$ an explicit rational-weighted rational $(t + d - 2)$-semidesign on $\mathcal{H}^1_0$ and $\mathcal{X}^1_1$ an explicit rational-weighted rational $(t + d - 1)$-semidesign on $\mathcal{H}^1_1$. Then,

$$\mathcal{Y}^1 \times \prod_{i=1}^{d-1} \mathcal{X}^1_{i \mod 2 \rightarrow i}$$

is an explicit spherical $t$-design on $S^d$.

Remark 16

- Designs above can be constructed over $\mathbb{Q}^{ab} \cap \mathbb{Q}$.
- Designs of arbitrary large size can be constructed.
Explicit spherical design

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