An explicit construction of spherical designs

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Spherical designs

Definition 1

A finite subset $X \subseteq \mathcal{S}^d$ is a spherical t-design provided that

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{\nu^d(S^d)} \int_{S^d} f \,\mathrm{d}\, \nu^d$$

for all $f \in \mathbb{R}[x_0, \ldots, x_d]_{\leq t}$, where ν^d is the spherical measure on $\mathcal{S}^d$.

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Related concept:

- Weighted design ($\mathcal{X} = (X, \mu_X)$).
- Rational design ($X \subseteq \mathbb{Q}^{d+1}$).
- Semidesign ($f \in \mathbb{R}[x_1, \ldots, x_d]_{\leq t}$).
- Rational-weighted rational semidesign.

Constructions of designs

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Problem 2

Are there rational spherical t-designs on $\mathcal{S}^d$ for all large d?

Structure of sphere and hemisphere as topological space

Let

$$H^d := \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} : \; x_0 > 0\}$$

be the d-dimensional open hemisphere.

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There exists a dominant open embedding of topological spaces

$$\begin{array}{ccccc} S^a & \times & H^b & \to & S^{a+b} \\ (x_0, \ldots, x_a) & \times & (y_0, \ldots, y_b) & \mapsto & (x_0 y_0, \ldots, x_a y_0, y_1 \ldots, y_b). \end{array}$$

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There exists an isomorphism of topological spaces

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Structure of sphere and hemisphere as measure space

Let $\mathcal{H}_s^d := (H^d, \nu_s^d)$ for certain measure ν_s^d on H^d. (The Radon-Nikodym derivative of ν_s^d with respect to the spherical measure ν^d is the polynomial $x_0 \mapsto x_0^s$.)

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$$\mathcal{S}^a \times \mathcal{H}_a^b \to \mathcal{S}^{a+b},$$

and an isomorphism of measure spaces

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Proposition 3

There exists a dominant open embedding of measure spaces

$$\mathcal{S}^1 \times \left(\mathcal{H}_1^1 \times \cdots \times \mathcal{H}_{d-1}^1\right) \to \mathcal{S}^d.$$

Sketch of an explicit construction of spherical designs

ERRS: explicit rational-weighted rational semidesign.

1. ERRS on $\mathcal{H}_0^1$.
2. ERRS on $\mathcal{H}_1^1$.
3. ERRS on $\mathcal{H}_s^1$.
4. ERRS on $\mathcal{H}_1^{d-1} \cong \mathcal{H}_1^1 \times \cdots \times \mathcal{H}_{d-1}^1$.
5. Explicit integer-weighted rational semidesign on $\mathcal{H}_1^{d-1}$.
6. Explicit design on $\mathcal{S}^1$.
7. Explicit design on $\mathcal{S}^d \sim \mathcal{S}^1 \times \mathcal{H}_1^{d-1}$.

Step 1. Rational-weighted rational semidesign on $\mathcal{H}_0^1$

Theorem 4

Choose (b_i, a_i) in $H^1 \cap \mathbb{Q}^2$ such that

$$\left| a_i - \sin \frac{(-t+2i+1)\pi}{2t} \right| < \frac{\pi^{2t}}{2^t t^{2t}}.$$

Then, $X := \{(b_i, a_i)\}$ is the support of a unique rational-weighted rational $(t-1)$-semidesign $\mathcal{X}_0^1 = (X, \mu_0^1)$ on $\mathcal{H}_0^1$. Moreover,

$$\mu_0^1(b_i, a_i) = \sum_{\text{even } j=0}^{t-1} \frac{e_{t-j-1}(a_1, \ldots, \hat{a}_i, \ldots, a_t)}{(j+1)\prod_{\substack{k \in [0,t-1]_{\mathbb{Z}} \\ k \neq i}} (a_k - a_i)},$$

where e_{t-j-1} is the $(t-j-1)$-th elementary symmetric polynomial.

Step 2. Rational-weighted rational semidesign on $\mathcal{H}_1^1$

Theorem 5

Assume that n is an odd integer multiple of even integer t and $n > t^{t/2}$. Choose (b_i, a_i) in $H^1 \cap \mathbb{Q}^2$ such that

$$\left| a_i - \frac{-n+1+2i}{n} \right| < \frac{t}{2n^4}.$$

Let $(b_i', a_i') := (b_j, a_j)$ where $j = \frac{(2i+1)n-t}{t}$. Then, $X := \{(b_i, a_i)\}$ is the support of a unique rational-weighted rational $(t-1)$-semidesign $\mathcal{X}_0^1 = (X, \mu_0^1)$ on $\mathcal{H}_0^1$ such that $\mu_0^1(b_i, a_i) = 1$ for $(b_i, a_i) \notin \{(b_i', a_i')\}$. Moreover,

$$\mu_0^1(b_i', a_i') = 1 + \sum_{j=0}^{t-1} (-1)^j \frac{e_{t-j-1}(a_1', \ldots, \hat{a_i'}, \ldots, a_t')}{\prod_{\substack{k \in [0,t-1]_{\mathbb{Z}} \\ k \neq i}} (a_k' - a_i')} \epsilon_{n,j}$$

where $\epsilon_{n,j} := \frac{1}{n} \sum_{i=0}^{n-1} a_i^j - \frac{1+(-1)^j}{2(j+1)}$.

Step 3. Rational-weighted rational semidesign on $\mathcal{H}_s^1$

Lemma 6

Let $\mathcal{X}_s^d = (X, \mu_s^d)$ *be a rational-weighted rational* $(t + \widetilde{s} - s)$*-semidesign on* $\mathcal{H}_s^d$*, where* $\widetilde{s} - s$ *is nonnegative even. Then,* $\mathcal{X}_{s\to\widetilde{s}}^d := (X, \mu_{s\to\widetilde{s}}^d)$ *is a rational-weighted rational t-semidesign on* $\mathcal{H}_{\widetilde{s}}^d$*, where*

$$\mu_{s\to\widetilde{s}}^d(x_0, \dots, x_d) := x_0^{\widetilde{s}-s}\mu_s^d(x_0, \dots, x_d).$$

Step 3. Rational-weighted rational semidesign on $\mathcal{H}_s^1$

Lemma 6

Let $\mathcal{X}_s^d = (X, \mu_s^d)$ be a rational-weighted rational $(t + \widetilde{s} - s)$-semidesign on $\mathcal{H}_s^d$, where $\widetilde{s} - s$ is nonnegative even. Then, $\mathcal{X}_{s \to \widetilde{s}}^d := (X, \mu_{s \to \widetilde{s}}^d)$ is a rational-weighted rational t-semidesign on $\mathcal{H}_{\widetilde{s}}^d$, where

$$\mu_{s \to \widetilde{s}}^d(x_0, \ldots, x_d) := x_0^{\widetilde{s} - s} \mu_s^d(x_0, \ldots, x_d).$$

Corollary 7

Let $\mathcal{X}_0^1$ be a rational-weighted rational $(t + s)$-semidesign on $\mathcal{H}_0^1$ and $\mathcal{X}_1^1$ a rational-weighted rational $(t + s - 1)$-semidesign on $\mathcal{H}_1^1$. Then, $\mathcal{X}_{i \bmod 2 \to i}^1$ is a rational-weighted rational t-semidesign on $\mathcal{H}_s^1$.

Step 4. Rational-weighted rational semidesign on $\mathcal{H}_1^{d-1}$

Lemma 8

Let $\mathcal{X}_0$ be a rational-weighted design on $\mathcal{Z}_0$ and $\mathcal{X}_1$ a rational-weighted design on $\mathcal{Z}_1$. Then, $\mathcal{X}_0 \times \mathcal{X}_1$ is a rational-weighted design on $\mathcal{Z}_0 \times \mathcal{Z}_1$.

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Corollary 9

For each $s \in [1, d-1]_{\mathbb{Z}}$, let $\mathcal{X}_s^1$ be a rational-weighted rational t-semidesign. Then,

$$\mathcal{X}_1^{d-1} := \mathcal{X}_1^1 \times \cdots \times \mathcal{X}_{d-1}^1$$

is a rational-weighted rational t-semidesign on $\mathcal{H}_1^{d-1} \cong \mathcal{H}_1^1 \times \cdots \times \mathcal{H}_{d-1}^1$.

Step 5. Integer-weighted rational semidesign on $\mathcal{H}_1^{d-1}$

Lemma 10

Let $\mathcal{X} = (X, \mu_X)$ be a rational-weighted design on $\mathcal{Z}$. Then, $\overline{\mathcal{X}} := (X, n_{\mathcal{X}}\mu_X)$ is an integer-weighted design on $\mathcal{Z}$, where

$$n_{\mathcal{X}} := \mathrm{lcm}_{x \in X} \text{ denominator of } \mu_X(x).$$

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Corollary 11

Let $\mathcal{X}_1^{d-1}$ *be a rational-weighted rational t-semidesign on* $\mathcal{H}_1^{d-1}$. *Then,* $\overline{\mathcal{X}_1^{d-1}}$ *is an integer-weighted rational t-semidesign on* $\mathcal{H}_1^{d-1}$.

Step 6. Designs on $\mathcal{S}^1$

Proposition 12

Let X be the vertices of a regular $(t+1)$-gon in $\mathcal{S}^1$. Then, X is a t-design on $\mathcal{S}^1$.

Step 7. Designs on $\mathcal{S}^d$

Lemma 13

Let $\mathcal{X}_0$ be a design on $\mathcal{Z}_0$ and $\mathcal{X}_1$ an integer-weighted design on $\mathcal{Z}_1$. Let $g : (0,1) \to \mathrm{Aut}(\mathcal{Z}_0)$ be a map such that $g(s)\,\mathcal{X}_0 \cap g(s')\,\mathcal{X}_0 = \emptyset$ for different $s, s' \in (0,1)$. Then,

$$\mathcal{X}_0 \rtimes \mathcal{X}_1 := \{(g(s_{x_1,i})x_0, x_1) : \; x_0 \in \mathcal{X}_0, x_1 \in \mathcal{X}_1, i \in [1, \mu_{X_1}(x_1)]_{\mathbb{Z}}\}$$

is a design on $\mathcal{Z}_0 \times \mathcal{Z}_1$, provided that $s_{x_1,i}$'s are distinct numbers in $(0,1)$.

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Corollary 14

Let $\mathcal{Y}^1$ be a design on $\mathcal{S}^1$ and $\overline{\mathcal{X}_1^{d-1}}$ an integer-weighted t-semidesign. Then,

$$\mathcal{Y}^1 \rtimes \overline{\mathcal{X}_1^{d-1}}$$

is a design on $\mathcal{S}^d$.

Explicit spherical design

Theorem 15

Let $\mathcal{Y}^1$ be an explicit t-design on $\mathcal{S}^1$, $\mathcal{X}_0^1$ an explicit rational-weighted rational $(t+d-2)$-semidesign on $\mathcal{H}_0^1$ and $\mathcal{X}_1^1$ an explicit rational-weighted rational $(t+d-1)$-semidesign on $\mathcal{H}_1^1$. Then,

$$\mathcal{Y}^1 \rtimes \overline{\prod_{i=1}^{d-1} \mathcal{X}^1_{i \bmod 2 \to i}}$$

is an explicit spherical t-design on $\mathcal{S}^d$.

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Remark 16

- Designs above can be constructed over $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{Q}$.
- Designs of arbitrary large size can be constructed.

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Thank you for your attention.