Tight Block Designs

Ziqing Xiang

University of Georgia

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Design

Definition 1

A F-design on X is a $Y \subseteq X$ such that

$$\int_X f \mathrm{d}\mu_X = \int_Y f \mathrm{d}\mu_Y,$$

for all $f \in F$.

X, Y: Both Hausdorff topological space and strictly positive probability measure space.
Y: A sub-topological space of X.
F: A linear space of measurable (on both X and Y) functions.

Block design

A t-(v, k, λ) design, t-design in short, consists of a set of

- points: a v-set V,
- blocks: a non-empty subset $\mathcal{B}$ of $\binom{V}{k}$,

such that for every $T \in \binom{V}{t}$,

$$\#\{B \in \mathcal{B} : \ T \subseteq B\} = \lambda > 0.$$

Example

0-design

Every subset $\mathcal{B}$ of $\binom{V}{k}$ is a 0-$(v, k, |\mathcal{B}|)$ design.

Example

Trivial design

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The set $\mathcal{B}$ is a t-$\left(v, k, \binom{v-t}{k-t}\right)$ design for all $t \leq k$.

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Partition

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The set $\mathcal{B}$ is a 1-$(v, 1, 1)$ design if and only if it is a partition.

Example

Finite projective plane

A projective plane consists of a set of points V, a set of lines $\mathcal{B}$ and an incidence relation between points and lines having:

- For every two different point, there exists a unique line incident with them;
- For every two different lines, there exists a unique point incident with them;
- There exists four points such that every line is incident with at most two of them.

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The pair $(V, \mathcal{B})$ is a finite projective plane if and only if it is a 2-$(n^2 + n + 1, n + 1, 1)$ design.

Example

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A Hadamard matrix of order $4n+4$ is equivalent to a 2-$(4n+3, 2n+1, n)$ up to isomorphism.

Existence of designs

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Theorem 3 (P. Keevash)

For sufficiently large v, if some necessary conditions are satisfied, then there exists t-(v, k, λ) design.

Lower bound for the size of designs

Theorem 4 (R. Fisher)

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Theorem 6 (D. Ray-Chaudhuri and R. Wilson.)

$|\mathcal{B}| \geq \binom{v}{e}$ *for 2e-designs when* $v \geq k+e$.

Sketch of the proof

- Construct the incidence matrix between $\binom{V}{e}$ vs $\mathcal{B}$.

$$M(E,B) = \begin{cases} 1, & E \subseteq B, \\ 0, & E \nsubseteq B. \end{cases}$$

- Show that the column space of the incidence is the whole space.
- $\binom{v}{e} \leq |\,\mathcal{B}\,|$.

Tight $2e$-design

A tight $2e$-design is a $2e$-design $\mathcal{B}$ with $|\mathcal{B}| = \binom{v}{e}$ and $v \geq k + e$.

A tight $2e$-design is non-trivial if $v > k + e$.

Classification of non-trivial tight $2e$-designs

- $e = 1$. Many. Classification is far from being complete.
- $e = 2$. H.Enomoto, N. Ito, R. Noda, A. Bremner. Only two, Witt 4-$(23, 7, 1)$ and Witt 4-$(23, 16, 52)$.
- $e = 3$. C. Peterson. None.
- $e \geq 5$. E. Bannai. Finitely many for each e.
- $e = 4$. E. Bannai. Finitely many.
- $5 \leq e \leq 9$. P. Dukes, J. Short-Gershman. None.
- $e = 4$. Z. Xiang. None.
- $e \geq 10$. ?

Intersection numbers

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Theorem 7 ((P. Delsarte;) D. Ray-Chaudhuri and R. Wilson)

For a tight 2e-(v, k, λ) design, the zeros of the polynomial $\Phi_e \in \mathbb{Q}[x]$ are intersection numbers of the design.

$$\Phi_e(x) := \sum_{i=0}^{e} (-1)^{e-i} \frac{\binom{v-e}{i}\binom{k-i}{e-i}\binom{k-i-1}{e-i}}{\binom{e}{i}} \binom{x}{i}.$$

Zeros of Φ_e

When

$$\frac{(v-k)^2k^2}{v^3}$$

is big, we can "use" the zeros of Φ_e to approximate the zeros of the Hermite polynomials $H_e \in \mathbb{Z}[x]$.

$H_0(x) = 1$, $H_1(x) = x$ and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$.

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Theorem 8 (E. Bannai)

There are only finitely many tight $2e$-design for each $e \geq 5$.

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Theorem 10 ((E. Bannai;) P. Dukes and J. Short-Gershman)

If there exists a non-trivial tight 8*-*(v, k, λ)*-design, then* (v, k) *is a zero of a polynomial* $f_4(v, k)$*.*

The polynomial f_4

$$
\begin{aligned}
f_4(v,\,k) = {} & \\
& -3408102864+1506333312k^2+974873344k^4-488998144k^6+62323584k^8-3309568k^{10}+65536k^{12}+9310949028v-1506333312kv- \\
& 4733985888k^2v-1949746688k^3v-1015706784k^4v+1466994432k^5v+511604992k^6v-249294336k^7v-49810560k^8v+16547840k^9v+ \\
& 1744896k^{10}v-393216k^{11}v-16384k^{12}v-11097146016v^2+4733985888kv^2+6922441360k^2v^2+2031413568k^3v^2-1428764528k^4v^2- \\
& 1534814976k^5v^2+209662720k^6v^2+199242240k^7v^2-21567744k^8v^2-8724480k^9v^2+786432k^{10}v^2+98304k^{11}v^2+7281931941v^3- \\
& 5947568016kv^3-4944873072k^2v^3+412538336k^3v^3+1856597696k^4v^3+243542016k^5v^3-293538048k^6v^3-13016064k^7v^3+ \\
& 17194752k^8v^3-327680k^9v^3-253952k^{10}v^3-2755473732v^4+3929166288kv^4+1497511456k^2v^4-1155170432k^3v^4-582955856k^4v^4+ \\
& 183266304k^5v^4+58253568k^6v^4-16432128k^7v^4-1102464k^8v^4+368640k^9v^4+544096980v^5-1459281552kv^5+28759472k^2v^5+ \\
& 469164960k^3v^5-7038496k^4v^5-59703552k^5v^5+6536960k^6v^5+2050560k^7v^5-328320k^8v^5-18769932v^6+293023248kv^6- \\
& 127930016k^2v^6-58917568k^3v^6+27050224k^4v^6+1258752k^5v^6-1642240k^6v^6+182784k^7v^6-14780538v^7-24513072kv^7+ \\
& 27560816k^2v^7-2875616k^3v^7-2296192k^4v^7+698880k^5v^7-61184k^6v^7+2961396v^8-764688kv^8-1582560k^2v^8+772608k^3v^8- \\
& 143664k^4v^8+10752k^5v^8-191952v^9+203472kv^9-52816k^2v^9+7520k^3v^9-640k^4v^9+972v^{10}-2352kv^{10}+336k^2v^{10}+45v^{11}
\end{aligned}
$$

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& 5947568016kv^3 - 4944873072k^2v^3 + 412538336k^3v^3 + 1856597696k^4v^3 + 243542016k^5v^3 - 293538048k^6v^3 - 13016064k^7v^3 + \\
& 17194752k^8v^3 - 327680k^9v^3 - 253952k^{10}v^3 - 2755473732v^4 + 3929166288kv^4 + 1497511456k^2v^4 - 1155170432k^3v^4 - 582955856k^4v^4 + \\
& 183266304k^5v^4 + 58253568k^6v^4 - 16432128k^7v^4 - 1102464k^8v^4 + 368640k^9v^4 + 544096980v^5 - 1459281552kv^5 + 28759472k^2v^5 + \\
& 469164960k^3v^5 - 7038496k^4v^5 - 59703552k^5v^5 + 6536960k^6v^5 + 2050560k^7v^5 - 328320k^8v^5 - 18769932v^6 + 293023248kv^6 - \\
& 127930016k^2v^6 - 58917568k^3v^6 + 27050224k^4v^6 + 1258752k^5v^6 - 1642240k^6v^6 + 182784k^7v^6 - 14780538v^7 - 24513072kv^7 + \\
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& 143664k^4v^8 + 10752k^5v^8 - 191952v^9 + 203472kv^9 - 52816k^2v^9 + 7520k^3v^9 - 640k^4v^9 + 972v^{10} - 2352kv^{10} + 336k^2v^{10} + 45v^{11}
\end{aligned}
$$

Theorem 11 (E. Bannai)

There are only finitely many tight 8-designs.

Tight 8-design

Theorem 12 (Z. Xiang)

There do not exist non-trivial tight 8-designs.

Sketch of the proof

For zeros (v, k) of f_4 with $2k \leq v \leq k^2$,

$$v = \frac{2}{1 - \sqrt[4]{\frac{3}{8}}} k + \frac{23}{500} \left(249 + 86\sqrt{6} + \sqrt{171312 + 70918\sqrt{6}} \right) + o(1).$$

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Find a $g(v, k)$ in the ring of integer-valued functions in v and k having

$$\lim_{v,k \to \infty} g(v, k) = \frac{9}{100} \left(6522 + 2808\sqrt{6} - \sqrt{56993328 + 24204417\sqrt{6}} \right).$$

An open problem

Fix positive integers c and n. Is the number of pairs of integers (a, b) satisfying the following conditions finite?

- $b \geq a + 2$.
- $\frac{a(a+1)}{b} \in \frac{1}{c}\mathbb{Z}$.
- $\frac{a(a+1)(a+2)}{b(b+1)} \in \frac{1}{c}\mathbb{Z}$.
- $\frac{a(a+1)(a+2)(a+3)}{b(b+1)(b+2)} \in \frac{1}{c}\mathbb{Z}$.
- $\dots$
- $\frac{a(a+1)\dots(a+n)}{b(b+1)\dots(b+n-1)} \in \frac{1}{c}\mathbb{Z}$.