Product of P-polynomial association schemes

Ziqing Xiang

Shanghai Jiao Tong University

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P-polynomial association scheme

Definition 1

P-polynomial association scheme $\mathcal{A} = (X, \{A_i\}_{0 \le i < d+1})$.

- A_i is X-by-X symmetric $(0,1)$ matrix.
- $\sum_i A_i = J$.
- $A_i = p_i(A_1)$, polynomials $p_i(x) = t_i x^i + o(x^i)$, $t_i > 0$.

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Distance-regular graph G.

- $V(G) = X$.
- $\mathrm{dist}(x, y) = i$, $A_i(x, y) = 1$.

Layers

$x_0 \in X$
i-th layer $X_i = \{x \mid \mathrm{dist}(x_0, x) = i\}$.
i-th valency $k_i = |X_i|$.

Shells form a partition of the ground set X.

Hamming scheme

Binary Hamming scheme $\mathcal{H}_v$

- Ground set $X = \mathbf{F}_2^v$.
- $\mathrm{dist}(x, y)$ is Hamming distance between x and y.
- $t_i = \frac{1}{i!}$, $p_i(x) = \frac{1}{i!}x^i + o(x^i)$.

Layer X_i consists of elements with exactly i ones.

Johnson scheme

Johnson scheme $\mathcal{J}_{v,k}$

- Ground set $X = X_k^{\mathcal{H}_v}$.
- $\mathrm{dist}(x, y)$ is half of Hamming distance between x and y.
- $t_i = \frac{1}{(i!)^2}$, $p_i(x) = \frac{1}{(i!)^2}x^i + o(x^i)$.

Design

Definition 2

(X, μ) is a measure space. V is a vector space consisting of measurable functions on X. $Y \subset X$ is a V-design if there exists a measure ν on Y satisfying:

$$\int_X f \mathrm{d}\mu = \int_Y f \mathrm{d}\nu$$

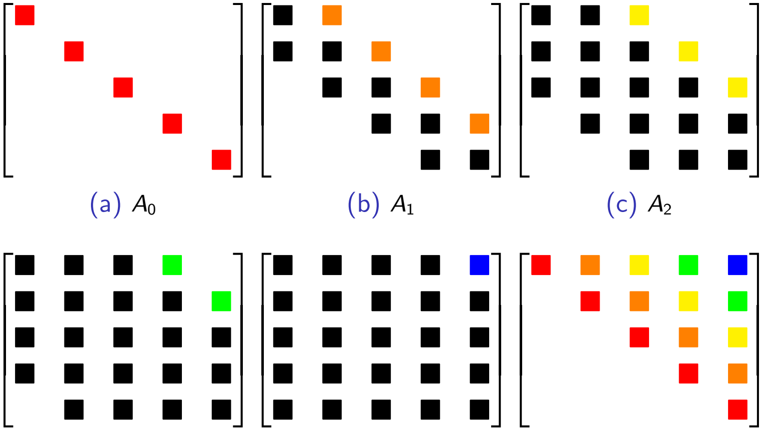
for every $f \in V$.

Some assumptions on measures are omitted.

A block matrix

$\mathcal{A}$ is a P-polynomial association scheme.

$$A(X_i, X_j) = A_{j-i}(X_i, X_j)$$



(a) A_0 (b) A_1 (c) A_2

(d) A_3 (e) A_4 (f) A

Relative design

$\mathcal{A}$ is a P-polynomial association scheme. $\mathrm{Pol}_t(X_S)$ is the row space of $A(\bigcup_{i\le t} X_i, X_S)$.

Definition 3

A relative t-design on layers $X_S = \bigcup_{s\in S} X_s$ is a subset Y of X_S such that

$$\sum_{x\in X_S} f(x)w_0(x) = \sum_{y\in Y} f(y)w_1(y)$$

for every $f \in \mathrm{Pol}_t(X_S)$.

Some assumptions on weights w_0 and w_1 are omitted.

Key property

$\mathcal{A}$ is a P-polynomial association scheme.
Let $\widetilde{A}(X_i, X_j) = t_{j-i}^{-1} A(X_i, X_j)$.

Lemma 4

$\widetilde{A}(X_i, X_j)\widetilde{A}(X_j, X_k) = \widetilde{A}(X_i, X_k)$

Proof.

$$
\begin{aligned}
&\widetilde{A}(X_i, X_j)\widetilde{A}(X_j, X_k) \\
=&A_1^{j-i}(X_i, X_j)A_1^{k-j}(X_j, X_k) \\
=&A_1^{j-i}(X_i, X)A_1^{k-j}(X, X_k) \\
=&A_1^{k-i}(X_i, X_k) \\
=&\widetilde{A}(X_i, X_k)
\end{aligned}
$$

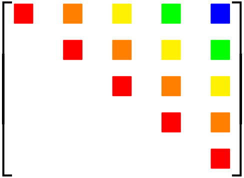


Figure : From http://dwenujang.blogspot.com

Cake

Definition 5

A cake $\mathcal{C}$ consists of:

- sets X_i, $0 \leq i < d+1$, which are called layers,
- a block $(0,1)$ matrix C,
- a positive sequence t_i,
- a modified block matrix $\widetilde{C}(X_i, X_j) = t_{j-i}^{-1} C(X_i, X_j)$,

and $\widetilde{C}(X_i, X_j)\widetilde{C}(X_j, X_k) = \widetilde{C}(X_i, X_k)$.

All P-polynomial association schemes are cakes.

Product cake

$\mathcal{A}$, $\mathcal{B}$ are cakes.

Definition 6

The product cake $\mathcal{C} = \mathcal{A} * \mathcal{B}$ is constructed as follow.

- $X_i^{\mathcal{C}} = X_i^{\mathcal{A}} \times X_i^{\mathcal{B}}$.
- $C(X_i, X_j) = A(X_i, X_j) \otimes B(X_i, X_j)$, namely $C = A * B$.
- $t_i^{\mathcal{C}} = t_i^{\mathcal{A}} t_i^{\mathcal{B}}$

$*$: Khatri-Rao product of two block matrices.

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Lemma 7

Johnson cake $\mathcal{J}_{v,k}$ is the product of Hamming cakes $\mathcal{H}_k$ and $\mathcal{H}_{v-k}$.

The product cake may not be an association scheme.

Cake design

$\mathcal{C}$ is a cake. $\mathrm{Pol}_t(X_S)$ is the row space of $C(\bigcup_{i\leq t} X_i, X_S)$.

Definition 8

A cake t-design on layers $X_S = \bigcup_{s\in S} X_s$ is a subset Y of X_S such that

$$\sum_{x\in X_S} f(x)w_0(x) = \sum_{y\in Y} f(y)w_1(y)$$

for every $f \in \mathrm{Pol}_t(X_S)$.

Some assumptions on weights w_0 and w_1 are omitted.

Lower bound of sizes of cake designs

One possible approach to establish the lower bound consists of three steps.

- Step 1: Cheesecake.
- Step 2: $\mathrm{Pol}_i(X_S)\mathrm{Pol}_j(X_S) \subseteq \mathrm{Pol}_k(X_S)$.
- Step 3: $C(X_i, X_{i+1})$ has full row rank.

Theorem 9

Y is a relative t-design of Johnson scheme on layers $X_S = \bigcup_{s\in S} X_s$, then under some assumption on elements in S, it holds

$$|Y| \geq \sum_{0\leq i<|S|} k_{a-i},$$

where $2a \leq t - |S| + 1$.

Step 1: Cheesecake

A matrix is almost strictly totally positive if every minor is nonnegative and the determinant of square submatrices with positive main diagonal is positive.

Definition 10

A cake is a cheesecake if the Toeplitz matrix $\mathbf{T}_{i,j} = t_{j-i}$ is almost strictly totally positive.

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Lemma 11

Under some assumption, a cake is a cheesecake if and only if the generating function

$$f(z) = \sum_i t_i z^i$$

has only real zeros.

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Corollary 12

Hamming cakes and Johnson cakes are cheesecakes.

Problem 13

Classify P-polynomial association schemes which are cheesecakes.

Step 2: $\mathrm{Pol}_i(X_S)$

$\mathcal{A}$, $\mathcal{B}$ are cakes, and $\mathcal{C} = \mathcal{A} * \mathcal{B}$.

Lemma 14

If C is a cheesecake,

$$\mathrm{Pol}_i^{\mathcal{A}}(X_S)\mathrm{Pol}_j^{\mathcal{A}}(X_S) \subseteq \mathrm{Pol}_k^{\mathcal{A}}(X_S),$$

and

$$\mathrm{Pol}_i^{\mathcal{B}}(X_S)\mathrm{Pol}_j^{\mathcal{B}}(X_S) \subseteq \mathrm{Pol}_k^{\mathcal{B}}(X_S),$$

then under some assumption on elements in S,

$$\mathrm{Pol}_i^{\mathcal{C}}(X_S)\mathrm{Pol}_j^{\mathcal{C}}(X_S) \subseteq \mathrm{Pol}_{k+|S|-1}^{\mathcal{C}}(X_S).$$

The bound $k + |S| - 1$ is sharp.

Step 2: $\mathrm{Pol}_i(X_S)$

Lemma 15

For Hamming cake, it holds

$$\mathrm{Pol}_i(X)\mathrm{Pol}_j(X) \subseteq \mathrm{Pol}_{i+j}(X).$$

Corollary 16

For Johnson cake, under some assumption on elements in S, it holds

$$\mathrm{Pol}_i(X_S)\mathrm{Pol}_j(X_S) \subseteq \mathrm{Pol}_{i+j+|S|-1}(X_S).$$

Step 3: $C(X_i, X_{i+1})$

Lemma 17

If $A(X_i, X_{i+1}) = |X_i|$ *and* $B(X_i, X_{i+1}) = |X_i|$*, then for* $\mathcal{C} = \mathcal{A} * \mathcal{B}$*,* $C(X_i, X_{i+1}) = |X_i|$*.*

Lemma 18

For Hamming cake $\mathcal{H}_v$*, it holds* $H(X_i, X_{i+1}) = |X_i|$ *for* $2i \leq v$*.*

Corollary 19

For Johnson cake $\mathcal{J}_{v,k}$*, it holds* $J(X_i, X_{i+1}) = |X_i|$ *for* $2i \leq \min\{k, v-k\}$*.*

Thanks for your attention.

Figure : From http://www.moevenpick-icecream.com/

Ziqing Xiang
Email: ZiqingXiang@gmail.com
Personal Homepage: http://ZiqingXiang.blogspot.com