Lit-only σ-game

Ziqing Xiang

Shanghai Jiao Tong University

June 24, 2013

This is joint work with Yaokun Wu.

Multigraph and graphs

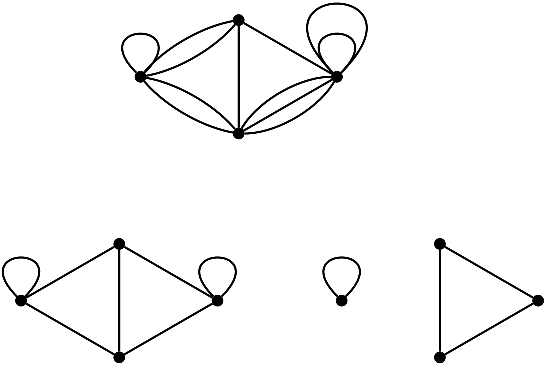

Figure : Two ways to view a multigraph as a graph.

Line graphs

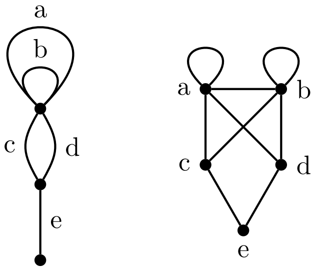

Figure : A multigraph and its line graph

Let graph G be the line graph of a multigraph H. The multigraph H is a root multigraph of the graph G.

Characterization of line graphs

A digraph is nonsingular if its adjacency matrix has full rank over binary field.

Theorem 1

For a loopless graph G, the following statements are equivalent.

- *The graph G is a line graph.*
- *The graph G does not contain any graph in a set of 32 forbidden graphs as induced subgraph.*

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For a loopless graph G, the following statements are equivalent.

- *The graph G is a line graph.*
- *The graph G does not contain any graph in a set of 32 forbidden graphs as induced subgraph.*
- *Every connected 6-vertex induced subgraph of G is a line graph.*
- *Every connected nonsingular 6-vertex induced subgraph of G is the line graph of a 7-vertex tree.*

The 32 forbidden graphs

There are 43 connected nonsingular 6-vertex graphs. They consist of the 32 forbidden graphs, and the 11 line graphs of 7-vertex trees.

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Problem 2

Is there any explanation using the root system of type E_6?

Critical subgraph

The adjacency matrix of a digraph D: $\mathbb{A}(D)$.
A critical subgraph of a graph G is a nonsingular induced subgraph H with $\operatorname{rank}\mathbb{A}(G) = \operatorname{rank}\mathbb{A}(H)$.

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Theorem 3

Critical subgraphs of a connected line graph are "almost" spanning trees in the root multigraph.

Existence of property preserving critical subgraphs

Theorem 4

For each of the following graph class $\mathcal{C}$, and every graph $G \in \mathcal{C}$, there exists graph $H \in \mathcal{C}$, such that H is a critical subgraph of G.

- *Connected graphs.*
- *Connected loopless line graphs.*
- *Connected line graphs with loops.*
- *Connected loopless non-line graphs.*
- *Connected non-line graphs with loops.*

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The existence of property preserving critical subgraphs allows us to extend a certain result for nonsingular graphs to the corresponding result for singular graphs.

Marble graph

Let $Q(G)$ be the Euler characteristic of G over $\mathbb{F}_2$, namely

$$Q(G) = |V(G)| - |E(G)| \pmod 2.$$

And for $x \in \mathbb{F}_2^{V(G)}$, set

$$Q_G(x) = Q(G[\mathrm{Supp}(x)]).$$

A marble graph is a loopless graph that satisfies

$$Q_G(x) = 0 \text{ whenever } \mathbb{A}(G)^\top x = \mathbf{0},$$

for $x \in \mathbb{F}_2^{V(G)}$

Equivalent definition

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$$Q_G(x) = 0 \text{ whenever } \mathbb{A}(G)^\top x = \mathbf{0},$$

for $x \in \mathbb{F}_2^{V(G)}$.

Theorem 5

A loopless graph G is a marble graph if and only if there exist the quadratic form q_G on the row space of $\mathbb{A}(G)$ that satisfies

$$q_G(\mathbb{A}(G)^\top x) = Q_G(x),$$

for $x \in \mathbb{F}_2^{V(G)}$.

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Is there any graph property that tells whether a graph is marble or not?

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Problem 6

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Theorem 7

Suppose connected loopless line graph G is the line graph of multigraph H. The graph G is a marble graph if and only if $|V(H)| \not\equiv 2 \pmod 4$.

Lit-only σ-game

Give a digraph D. For every $v \in V(D)$, construct a map $\mathcal{T}_v \in \mathrm{End}(\mathbb{F}_2^{V(D)})$ by setting

$$\mathcal{T}_v(x)(w) = \begin{cases} x(w), & vw \notin A(D), \\ x(w) + x(v), & vw \in A(D). \end{cases}$$

The map $\mathcal{T}_v$ is a transvection when v is not a loop, and is a projection when v is a loop.

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The phase space of the lit-only σ-game on a digraph D is the digraph Γ with:

- $V(\Gamma) = \mathbb{F}_2^{V(D)}$;
- $A(\Gamma) = \{(x, \mathcal{T}_v(x)) \mid x \in V(\Gamma), v \in V(D)\}$.

Lit-only group

The lit-only group of a loopless digraph D, $\mathrm{LOG}(D)$, is the multiplicative group generated by $\mathcal{T}_v$ where $v \in V(D)$.

Lit-only group

The lit-only group of a loopless digraph D, LOG(D), is the multiplicative group generated by $\mathcal{T}_v$ where $v \in V(D)$.

The lit-only group determines the phase space, and vice versa.

Classification of lit-only group

Let D be a strongly connected loopless digraph, let $V = \mathbb{F}_2^{V(D)}$ and W be row space of $\mathbb{A}(D)$.

Theorem 8

- *D is the line graph of multigraph H:*
  $\mathrm{LOG}(D) \cong \mathrm{Sym}_{V(H)} \ltimes W^{\dim V - \dim W - c}$,
  where $c = 1 + |V(H)| \pmod 2$.
- *D is a non-line marble graph:*
  $\mathrm{LOG}(D) \cong \mathrm{O}(W, q_D) \ltimes W^{\dim V - \dim W}$.
- *D is a non-line non-marble graph:*
  $\mathrm{LOG}(D) \cong \mathrm{Sp}(W, \omega_D) \ltimes W^{\dim V - \dim W - 1}$.

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Conjecture 9

Non-symmetric digraph:
$\mathrm{LOG}(D) \cong \mathrm{SL}(W) \ltimes W^{\dim V - \dim W}$.

Conjecture 10

Let D be a strongly connected loopless digraph, let $V = \mathbb{F}_2^{V(D)}$ and W be row space of $\mathbb{A}(D)$. The group $\mathrm{LOG}(D)$ is isomorphic to one of the following.

- $\mathrm{Sym}_{\dim W+1} \ltimes W^{\dim V - \dim W}$
- $\mathrm{Sym}_{\dim W+2} \ltimes W^{\dim V - \dim W - 1}$
- $\mathrm{O}^{-}(W) \ltimes W^{\dim V - \dim W}$
- $\mathrm{O}^{+}(W) \ltimes W^{\dim V - \dim W}$
- $\mathrm{Sp}(W) \ltimes W^{\dim V - \dim W - 1}$
- $\mathrm{SL}(W) \ltimes W^{\dim V - \dim W}$

Problem 11

Is there any connection with the classification of a certain type of Lie algebra?

Lit-only σ-game on graphs with loops

Forbidden subgraph characterization for connected line graphs with loops is found.
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Problem 12

Give a classification of the monoid generated by $\mathcal{T}_v$.

Problem 13

Is there any connection with Hecke algebra?

Lit-only σ-game on digraphs

- Line digraphs.

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- Line digraphs.
- Critical subdigraphs.

Problem 14

Give a characterization of strongly connected digraphs that have strongly connected critical subdigraphs.

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Problem 14

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- Ear decomposition.

Only partial results.

Lit-only σ-game on vector bundle

Let V be a vector space and D be a digraph. For each arc $vw \in A(D)$, we put an automorphism $\phi_{vw} \in \mathrm{Aut}(V)$.
For each $v \in V(D)$, construct map $\mathcal{T}_v \in \mathrm{End}(V^{V(D)})$ by setting

$$\mathcal{T}_v(\mathbf{x})(w) = \begin{cases} \mathbf{x}(w), & vw \notin A(D), \\ \mathbf{x}(w) + \phi_{vw}(\mathbf{x}(v)), & vw \in A(D), \end{cases}$$

for $\mathbf{x} \in V^{V(D)}$.
The phase space is the digraph Γ with

- $V(\Gamma) = V^{V(D)}$;
- $A(\Gamma) = \{(\mathbf{x}, \mathcal{T}_v(\mathbf{x})) \mid \mathbf{x} \in V(\Gamma), v \in V(D)\}$.

The lit-only sigma game on a digraph is the game on a vector bundle with $V = \mathbb{F}_2$.

Thanks for your attention.

Ziqing Xiang
Email: ZiqingXiang@gmail.com
Personal Homepage: http://ZiqingXiang.blogspot.com