Nonexistence of nontrivial tight 8-design

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Outline

- t-design
- Fisher type lower bound
- Tight $2s$-design

t-design

Denote set $\{1, \ldots, v\}$ by $[v]$.
$\mathcal{B}$ is a t-(v, k, λ) design, t-design in short, if

- $\mathcal{B}$ is a subset of $\binom{[v]}{k}$,
- for any $S \in \binom{[v]}{t}$, $\#\{B \in \mathcal{B} : S \subseteq B\} = \lambda > 0$.

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$\mathcal{B} = \binom{[v]}{k}$ is a trivial design.

Examples

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- $t = 2$. Fano plane.

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A Steiner system is a t-design with $\lambda = 1$ and $t \geq 2$.

Parameter

Let $\mathcal{B}$ be a t-(v, k, λ) design. Then for any subsets $I, J \subseteq [v]$ with $I \cap J = \emptyset$, $|I| + |J| \leq t$,

$$\#\{B \in \mathcal{B} : I \subset B, J \subset \overline{B}\} = \lambda \frac{\binom{v-|I|-|J|}{k-|I|}}{\binom{v-t}{k-t}}.$$

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For t-(v, k, λ) design $\mathcal{B}$,

- $|\mathcal{B}| = \lambda \binom{v}{t} / \binom{k}{t}$;
- $\mathcal{B}$ is a $(t-1)$-design.
- the complementary design $\widetilde{\mathcal{B}} = \{\overline{B} : B \in \mathcal{B}\}$ is a t-design.

Fisher type lower bound

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Dijen K. Ray-Chaudhuri & Richard M. Wilson. $|\mathcal{B}| \geq \binom{v}{s}$ for $2s$-designs.

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M has full row rank.

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Assume $v \geq 2k$.

Classification of tight $2s$-designs

- $s = 2$. Hikoe Enomoto. Noboru Ito. Ryuzaburo Noda. Andrew Bremner. Et al. Witt 4-$(23, 7, 1)$. Witt 4-$(23, 16, 52)$.

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Intersection numbers

For a tight $2s$-(v,k,λ) design, zeros of the following polynomial are intersection numbers.

$$\Phi_s(x) = \sum_{i=0}^{s} (-1)^i \frac{\binom{v-s}{i}\binom{k-i}{s-i}\binom{k-i-1}{s-i}}{\binom{s}{i}} \binom{x}{i}.$$

Hermite polynomials

$H_0(x) = 1$, $H_1(x) = x$, and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$

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- $s = 4$. Eiichi Bannai. Finitely many.

A thirteen degree polynomial

$$
\begin{aligned}
f(k, v) = {} & \\
& -3408102864 + 1506333312k^2 + 974873344k^4 - 488998144k^6 + 62323584k^8 - 3309568k^{10} + 65536k^{12} + 9310949028v - 1506333312kv - \\
& 4733985888k^2v - 1949746688k^3v - 1015706784k^4v + 1466994432k^5v + 511604992k^6v - 249294336k^7v - 49810560k^8v + 16547840k^9v + \\
& 1744896k^{10}v - 393216k^{11}v - 16384k^{12}v - 11097146016v^2 + 4733985888kv^2 + 6922441360k^2v^2 + 2031413568k^3v^2 - 1428764528k^4v^2 - \\
& 1534814976k^5v^2 + 209662720k^6v^2 + 199242240k^7v^2 - 21567744k^8v^2 - 8724480k^9v^2 + 786432k^{10}v^2 + 98304k^{11}v^2 + 7281931941v^3 - \\
& 5947568016kv^3 - 4944873072k^2v^3 + 412538336k^3v^3 + 1856597696k^4v^3 + 243542016k^5v^3 - 293538048k^6v^3 - 13016064k^7v^3 + \\
& 17194752k^8v^3 - 327680k^9v^3 - 253952k^{10}v^3 - 2755473732v^4 + 3929166288kv^4 + 1497511456k^2v^4 - 1155170432k^3v^4 - 582955856k^4v^4 + \\
& 183266304k^5v^4 + 58253568k^6v^4 - 16432128k^7v^4 - 1102464k^8v^4 + 368640k^9v^4 + 544096980v^5 - 1459281552kv^5 + 28759472k^2v^5 + \\
& 469164960k^3v^5 - 7038496k^4v^5 - 59703552k^5v^5 + 6536960k^6v^5 + 2050560k^7v^5 - 328320k^8v^5 - 18769932v^6 + 293023248kv^6 - \\
& 127930016k^2v^6 - 58917568k^3v^6 + 27050224k^4v^6 + 1258752k^5v^6 - 1642240k^6v^6 + 182784k^7v^6 - 14780538v^7 - 24513072kv^7 + \\
& 27560816k^2v^7 - 2875616k^3v^7 - 2296192k^4v^7 + 698880k^5v^7 - 61184k^6v^7 + 2961396v^8 - 764688kv^8 - 1582560k^2v^8 + 772608k^3v^8 - \\
& 143664k^4v^8 + 10752k^5v^8 - 191952v^9 + 203472kv^9 - 52816k^2v^9 + 7520k^3v^9 - 640k^4v^9 + 972v^{10} - 2352kv^{10} + 336k^2v^{10} + 45v^{11}
\end{aligned}
$$

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β_0

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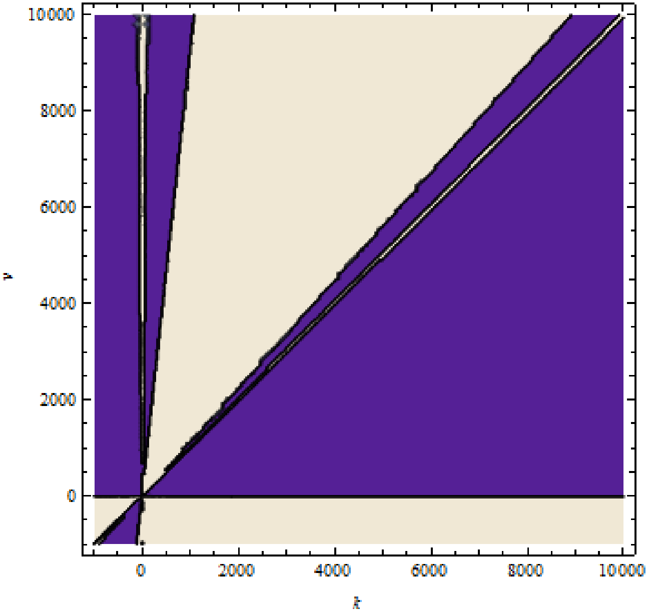
Figure



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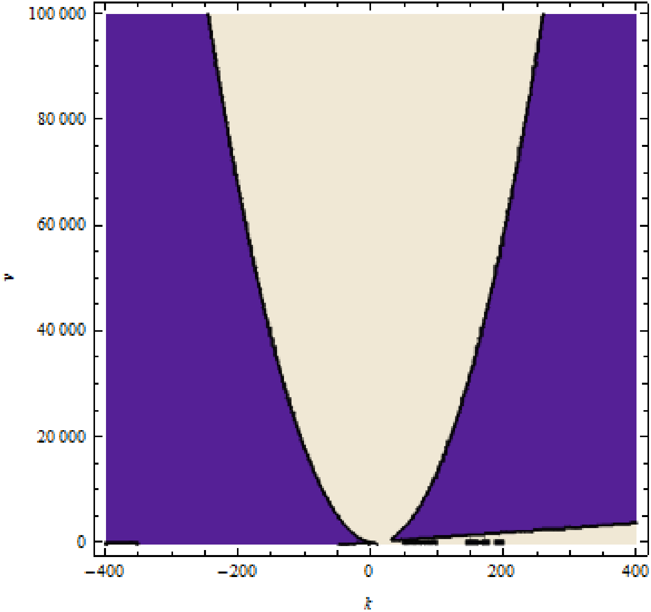

Zeros

$$v = \frac{2}{1 - \sqrt[4]{\frac{3}{8}}} k + \frac{23}{500} \left(249 + 86\sqrt{6} + \sqrt{171312 + 70918\sqrt{6}} \right) + o(1)$$

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$$g(v, k) \in \mathbf{Z}$$

$$\lim_{v,k \to \infty} g(v, k) = \frac{9}{100} \left(6522 + 2808\sqrt{6} - \sqrt{56993328 + 24204417\sqrt{6}} \right)$$

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- $s \geq 10$. ?

Thanks for your attention.